A New and Comprehensive

SYSTEM

OF

Mathematical Institutions,

Agreeable to the

PRESENT STATE

OF THE

Newtonian Mathesis.

VOL. I.

Containing the Institutes, or Principles of

I. ARITHMETIC, Vulgar and Decimal.
II. LOGARITHMS.
III. ALGEBRA.
IV. GEOMETRY.
V. Of Plain TRIGONOMETRY.
VI. Of CONIC SECTIONS.
VII. FLUXIONS.

By Benjamin Martin.

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MDCCLIX.

~~Math 357.59~~
P 193.4*

TO

HIS ROYAL HIGHNESS

GEORGE AUGUSTUS,

PRINCE of WALES.

SIR,

IN every Institution of Government, founded on just Principles, the Felicity of the Prince and the People must necessarily be reciprocal, as it results equally from a wise and benevolent Exercise of Dominion in the one, and good Sense, and rational Subjection, in the other: This is a Truth verified by a Reflexion on the Fate of Nations in general, but more especially confirmed by that of our own, as well in the last as present Reign, in which the Blessings of Nature have been accumulated on the *British* Nation almost to Profusion, and we may say, with more Justice than could be said of them in former Times, that the People would, if possible, be too happy were they truly sensible of their present blissful Situation in its utmost Extent. But to do our-

ourſelves Juſtice, it muſt be allowed, we are not altogether inſenſible of our Happineſs; for I, with the higheſt Pleaſure, aſſure Your Royal Highneſs, that in more than half the great Towns in *England*, and among all Ranks of People, I have been a conſtant Eye and Ear-Witneſs of their univerſal Joy and Satisfaction with their preſent Condition; of their extraordinary Eſteem and Regard for the Perſon, Title and Government of his preſent Moſt Gracious Majeſty, and their moſt exalted Hopes, and higheſt Confidence in their future Sovereign: This good Diſpoſition, permit me, Great Sir, to ſay, is the natural Conſequence of encouraging the Studies of uſeful Arts and Sciences, Learning and Humanity; for theſe furniſh the Prince with Notions and Principles of Wiſdom, Religion, Virtue, and Liberty, and ſecure the People from the Attacks of Ignorance, Barbarity, Superſtition and Impoſture; and in every Perſon they produce a rational and noble Propenſity towards promoting the general Good of the Community, and the Promulgation of the Sciences among all Ranks and Orders of Men, and inculcate on their Minds Principles that will not fail to render them good Subjects: As this is the profeſſed Deſign of theſe Papers, I humby preſume they will be acceptable to Your Royal Highneſs, and ſhall for ever eſteem it the higheſt Honour that I am permitted to offer them to Your Highneſs's Inſpection. That Heaven may preſerve his preſent Sacred Majeſty to the lateſt deſirable Period of Life, and then Your Highneſs aſcend the *Britiſh* Throne and long reign the happieſt, as well as the greateſt, Monarch of the World, is the inceſſant Prayer of,

SIR,

Your Royal Highneſs's

Moſt dutiful, devoted,

And obedient humble Servant,

B. MARTIN.

ON THE USEFULNESS OF MATHEMATICAL LITERATURE:

A POEM.

—— —— —— —— —— Ni
Posces ante diem librum cum lumine, si non
Intendes animum studiis et rebus honestis;
Invidiâ vel amore vigil torquebere. ——

HOR. Epist. 2. Lib. I.

HAIL! heav'nly Science, *bright*, seraphic Truth,
My Soul's Delight, the Labour of my Youth;
O! prove propitious to my artless Lays,
Which I devote with Pleasure to thy Praise.
When dread Jehovah *form'd the Worlds above,*
Assign'd their Posts, and bid the System *move*;
The great, omnipotent, eternal Cause,
Commanded thee to regulate their Laws:
(Obedience waits the Word) thy Pow'r to prove,
In perfect Music at thy Will they move.
In Afric's *Plains where antient* Nilus' *Flood,*
Draws up the Marks, and stains the Lands with Blood;

To

To teach weak Mortals, Property to ſcan,
Down came Gëometry *and form'd a Plan.*
Mankind, poſſeſs'd of this celeſtial Art,
To diſtant Climes its Influence impart;
Conſtrain the Woods o'er Ocean's *Face to rove,*
And mete the ſpangled Firmament above;
Succeeding Heads, the Properties deviſe
Of Lines on Lines, that meet our wond'ring Eyes;
All Nature ſeem'd ſubmiſſive to the Skill,
And form'd obedient to the human Will.
The Property *laid down, the* Proof *was clear,*
But how to make new Properties *appear*
Was known to few,—(the Joys that Genius brings
Are far ſuperior to all earthly Things;
But deep Invention *ſeldom wills to ſtray*
Amongſt the Proud, the Buſy, or the Gay)
An Art, by which deep Secrets are reveal'd,
Some Antients knew, but carefully conceal'd;
How far by this they clear'd Invention's *Way*
Is hard for modern Geometers *to ſay:*
Its Uſe in Lines in ſome Degree they knew,
As Theon's *Reas'nings evidently ſhew.*
Indeed, what might not Greeks *effect, in Days,*
When Monarchs crown'd Philoſophers with Bays;
When ſolid Learning flouriſh'd thro' the State,
And to be wiſe, was reckon'd truly great:
No foreign Tongue perplex'd their happy Youth,
They taught them Virtue, *and her Siſter* Truth;
Soul-moving Eloquence, ſound moral Rules,
And thou, O Science! *grac'd their public Schools.*
But Sages hold, there's in all mundane Things,
The Cauſe, from which their Diſſolution ſprings;
So Greece *had kept her Learning, and her Fame;*
Had not falſe Sophiſtry *aſſum'd thy Name:*
O'er ſolid Reaſon, rul'd without controul;
And groſs Stupidity hood-wink'd the Soul:
Then, Truth's *bright Eſſence fled the human Race,*
And Learning gone, Barbarity took Place.

But

But to descend thro' dark, unwelcome Times,
To happier Days, in these vesperian Climes;
Where Truth's *fair Goddess animated Man,*
For Science self we in a Newton *scan!*
Amazing Genius! Whose prolific Thought,
Nature unveil'd, and deep Fluxonia *taught:*
An Art, that makes the hardest, antique Rules,
But boyish Problems, in our modern Schools.
These neighb'ring Nations to thee, Science, *owe,*
The new-found Regions, whence their Riches flow;
Who taught Columbus *Ocean's Face to sweep,*
Or kindred Cabots *how to plow the Deep,*
But thou alone?—And from this Source there springs,
The Pow'r, and Grandeur of European *Kings.*
Mankind, by thee, to farthest India *roam,*
Whence orient Pearls, Gems, Silks, and Spices come:
Where would Britannia's *Ships, her Commerce be;*
Her Pow'r, her All———O Science, *but for thee?*
The meanest Rustic craves thy powerful Aid,
To mete the Labour of his toilsome Spade:
Those grand Machines whence Manufactures flow,
(Employ for Thousands) their Invention owe
To thee:——These Engines Collieries demand,
By Art obstetric form'd, fair Science plann'd:
Works to force Water, 'Gins to conquer Fire,
Air-mills, and aqueous; all thy Aid require:
Clocks, noting Time's incessant circling Course,
Ballistic Engines all-destroying Force,
The Orrery sublime, th' Armillary Sphere,
Glasses, that weigh the pressing, ambient Air,
Sun-dials, optic Instruments benign,
Perspective Drawings, and the Globes are thine.
Except religious Comforts, Naught below
Equals the Bliss that Science' *Vot'ries know:*
These happy Mortals, Pallas, *by thy Pow'r,*
To distant Worlds in Contemplation tour;
What mental Bliss to view the Order giv'n,
To those bright Orbs; the Ornament of Heav'n!

Not

Not fond Caresses, nor each charming Grace,
That plays on blooming Celia's *matchless Face;*
Enamel'd Meadows, nor luxuriant Fields,
Give half the Pleasure, Truth's *fair Goddess yields.*
Winter, *will strip the happiest sylvan Shades;*
Old Age, or Sickness, spring-like Beauty fades;
But those refin'd Enjoyments Science *brings,*
Like 'liv'ning Sol, *are everlasting Things:*
Compleat the reas'ning Faculties, and save
The Soul that Trouble, when beyond the Grave.
Think! then, ye Mortals, of your murder'd Hours,
In hurrying Cities, or in rural Bow'rs;
Cast off your Indolence, your Minds careen,
And learn fair Science *from her* MAGAZINE.

CHARLES WILDBORE.

INSTI-

INSTITUTIONS OF ARITHMETIC.

The Introduction.

1. MATHESIS is the Science or Doctrine of *Quantity*, whose various Relations and Affections it contemplates, and gives Rules for making an *Estimate*, or *Computation* thereof; which it expresses in different Ways, and by various Kinds of Symbols or Characters, which is called *Notation.*

2. *Quantity* is every Thing which we can conceive to have any Magnitude or Parts, or, properly speaking, it is any Thing concerning which we can ask the Question, *how much?* Or, *how great?* It is often distinguished into *continued* and *discrete.*

3. *Continued Quantity* is that whose Parts are all contiguous, or adhere together, and make but *one Whole*, as a *Shilling*, a *Stone*, a *Sheep*, &c. this is the Subject of that Part of *Mathesis* which is called *Geometry.*

4. *Discrete Quantity* is that whose Parts are not continuous, but separate from each other, and make what Logicians call a *collective Idea*, or *Whole*; thus *one Pound* consists of 20 *Shillings*; a *Flock* of many Sheep, *&c.* this is the Subject of the other Part of Mathesis called *Arithmetic.*

5. Notation is of three Kinds, *viz. Numerical*, *Specious*, and *Linear*; *Numerical Notation* is the Representation of Quantity or Quantities by those Characters we call *Numbers* or *Digits*, which are in all ten, *viz.* 0, Cypher; 1, One; 2, Two; 3, Three; 4, Four; 5, Five; 6, Six; 7, Seven; 8, Eight; 9, Nine. These are used in common Arithmetic.

6. *Specious Notation* is that wherein *Species*, or *Letters*, are made uſe of to repreſent Quantities, as the Letters of the Alphabet, *a*, *b*, *c*, or *A*, *B*, *C*, *&c.* as in *Algebra*. Or the ſame Letters with Points over them; thus, $\dot{x}$, $\dot{y}$, $\dot{z}$, in that Part of the Science called *Fluxions*.

7. *Linear Notation*, is the Repreſentation of Quantities by *Lines*, and *Figures* compoſed of *Lines*, as is done in all the Parts of *common Geometry*.

8. The Quantities conſidered in *Arithmetic* are called *Numbers*, of which there are two Sorts, *whole* and *broken*, which are otherwiſe called *Integers* and *Fractions*. The leaſt whole Number is *Unity*, or 1, One; that is, any *one Thing* is called an *Unite*; and *Nothing*, or *Nullity*, is repreſented by the Cypher 0.

9. A Number of Units under ten, is repreſented by a ſingle Digit, as 2, 5, 7, *&c.* but *ten Units* are deſigned by the firſt Digit 1, with a Cypher annexed to the Right Hand, thus, 10. And as an Unit is made Ten by one Cypher annexed, ſo it is made *Ten Times Ten*, or *an Hundred*, by two annexed Cyphers, *viz.* 100; ſo another Cypher makes *Ten Hundred*, or *One Thouſand*, *viz.* 1000, and ſo on, as in the following Table.

1	Unit
10	Ten
100	Hundred
1000	Thouſand
10000	Ten Thouſand
100000	Hundred Thouſand
1000000	Million
10000000	Ten Millions
100000000	Hundred Millions
1000000000	Thouſand Millions.

10. If the Number conſiſts not of even Tens, then ſuch a Digit is annexed to the Unit as will expreſs the ſaid Number; thus Seventeen is expreſſed by 17. Alſo *twice Ten*, or *Twenty*, is expreſſed by the Figure 2 and a Cypher, thus, 20; and *Thirty* by 30; *Forty* by 40; and ſo on to an Hundred. The intermediate Numbers are alſo expreſſed by annexing proper Digits in Place of the Cypher; thus 25 is Twenty-five; 37 Thirty-ſeven, *&c.*

11. From

11. From hence we obtain the Method of *Enumeration*, or expreſſing the Number of Quantities contained in any given *Sum*, as ſhewn below.

Numeration Table.

Hundreds of Millions	Tens of Millions	Millions	Hund. of Thouſ.	Tens of Thouſ.	Thouſands	Hundreds	Tens	Units	
								1	One
							2	1	Twenty-one
						3	2	1	Three Hundred, Twenty-one
					4	3	2	1	4 Thouſand, 3 Hundred, 21
				5	4	3	2	1	54 Thouſand, 3 Hundred, 21
			6	5	4	3	2	1	6Hundred, 54Thouſand, 3Hundred, 21
		7	6	5	4	3	2	1	Seven Millions, 654 Thouſand, 321
	8	7	6	5	4	3	2	1	87 Million, 654 Thouſand, 321
9	8	7	6	5	4	3	2	1	987 Million, 654 Thouſand, 321.

Here it is plain, that in order to numerate the Figures in any Sum, you have only need to mention firſt each Figure, and then the Place in which it ſtands, according to its Name of Valuation in the Table, in the ſame Manner as you ſee done for each Sum, or Line of Figures in the Table on the Right Hand Side. Thus for Inſtance, the Sum 850243 you read or value thus, 8 Hundred, 50 Thouſand, 9 Hundred, 43; or thus at twice, 850 Thouſand 943; and ſo the Sum 406528035 is thus read, 406 Million, 528 Thouſand, 35; and ſo of others.

12. As an Unit may have its Value encreaſed ten Times, by annexing a Cypher to the Right Hand, ſo its Value is diminiſhed in a *ten-fold Proportion* by prefixing Cyphers thereto; thus

0,1 is one Tenth
0,01 is one Hundredth } of an Unit.
0,001 is one Thouſandth

Thus alſo { 0,7 is Seven Tenths.
0,53 is Fifty-three Parts of an Hundred.
0,375 is Three Hundred Seventy-five Parts of a Thouſand.

13. In this Caſe, the Cypher on the Left, cut off with a Comma (,) ſtands in the Unit's Place, and ſhews the Number does not amount to Unity, but is a certain Number of ſuch Parts as the Unit contains 10, or 100, or 1000, &c. and theſe Parts are

expressed by the Figures on the Right Hand of the Comma. This Kind of Notation of the Parts of a broken or divided Unit is called *Decimal*, (from *Decem*, Ten) and those Parts of Unity are called *Decimal Numbers*, or *Decimal Fractions*.

14. Sometimes a Number consists of *Integers* and *Decimals* together, and is then called a *mixed Number*; thus 7,3 is Seven and three Tenths; 84,53 is eighty four and fifty-three Parts of an Hundred of another; and so of others. That Part of the Science which treats of those Numbers is called *Decimal Arithmetic*.

15. If Unity be divided in any other than a *ten-fold Proportion*, then another Species of Computation will ensue; thus in Astronomy we divide a Degree into 60 equal Parts or Minutes; these Minutes are each divided into 60 *Seconds*; each Second into 60 *Thirds*, and so on to *Fourths*, *Fifths*, &c. And they are thus denoted, *viz.* $35^{\circ} : 47' : 31'' : 23'''$; *Thirty-five Degrees, Forty-seven Minutes, Thirty-one Seconds, Twenty-three Thirds.*— The Rules for managing these Numbers is called *Sexagenary*, or *Sexagessimal Arithmetic*.

16. It frequently happens that we are obliged to divide an Unit indefinitely, or into any Number of Parts as Occasion requires for comparing a Part with the whole Unit, in Parts of such a Division: In this Case, the Way to express such a Fraction, is to place the Unit divided into its whole Number of Parts below a Line, and the Parts of the Unit which are given above it; thus $\frac{3}{4}$ is three Parts of such as the Unit contains four of; and $\frac{5}{13}$ is five Thirteenths of the whole Unit. These are called *Vulgar Fractions*.

17. A Vulgar Fraction is said to be *pure*, when it consists only of fractional Parts, as $\frac{1}{2}$, $\frac{13}{43}$, $\frac{23}{751}$, *&c.* and *mixed*, when joined with Integers, as $5\frac{1}{7}$, $23\frac{4}{9}$, $1\frac{1}{751}$, *&c.*

18. The Number placed below the Line, is called the *Denominator* of the Fraction, because it denominates the Fraction, or Number of Parts into which Unity is broken or divided; and the Number above the Line is called the *Numerator*, because it enumerates or shews how many of those Parts make the Fraction proposed.

19. The Fraction is said to be *proper*, when the Numerator is less than the Denominator, as $\frac{3}{4}$; but *improper*, when the contrary happens, as $\frac{4}{3}$, $\frac{13}{9}$, *&c.*

20. When

20. When any two Quantities are compared together, to observe the Relation of their *Magnitude*, ſuch Compariſon is called a *Ratio*; and is thus expreſſed, $a : b$; of this *Ratio*, the firſt Term (a) is called the *Antecedent*, and latter (b) the *Conſequent*.

21. When any two Quantities have the ſame *Ratio* with any other two, it is denoted by this Character : : thus $a : b :: c : d$; the Quantity (a) is to (b) as (c) is to (d); which are therefore ſaid to be *analogous* or *proportionate*; and ſuch a Compariſon, or Expreſſion, is called *Analogy*, or *Proportion*.

22. When any Calculation is to be made, it is done either by *Addition*, *Subtraction*, *Multiplication*, or *Diviſion* of Quantities; which four fundamental Rules are called the *Algorithm* of Quantities, and which we now proceed to explain.

Characters for Abbreviation explained,

viz. { $+$, $-$, $\times$, $\div$, $=$ } ſignifies { More; as $3+4$, is 3 added to 4.
Leſs; as $4-3$, is 3 taken from 4.
Multiplied by; as 3×4, is 3 multiplied by 4.
Divided by; as $3 \div 4$, is 3 divided by 4, or $\frac{3}{4}$.
Equal to; as $a = 4$, is a equal to 4.

CHAPTER I.

ADDITION *of* INTEGERS, *or* WHOLE NUMBERS.

23. ADDITION of Numbers conſiſts in adding together all the Units contained in ſeveral particular Numbers, properly diſpoſed, into one *Sum*, *aggregate* or *total*, expreſſing the Value of all together. And this is performed in the following Manner, *viz.*

24. Let the ſeveral particular Sums, or Numbers, be diſpoſed one under another in ſuch a Manner that the Place of Units, Tens, Hundreds, &c. in each may conſtitute a perpendicular Column of Figures; thus let it be required to add together the Numbers 57, 762, 5389, 97615; in order to do this they muſt firſt be diſpoſed thus,

$$\begin{array}{r} 57 \\ 762 \\ 5389 \\ 97615 \\ \hline \end{array}$$

25. The

25. The Numbers placed, as above, you proceed to add together by the following

RULE,

Reckon up all the Digits in the first, or Right-Hand Column, and observe, for every Ten to carry one to the Place of Tens in the second Column, setting down the remaining Digits under the first Column of Units: Thus $5+9+2+7=23$, which is thus expressed, *five* more *nine* more *two* more *seven* is *equal* to *twenty three*, in which Sum there are *two Tens* and 3 over; you must then set down the 3, and carry two to the next Place of Tens, and proceed as before; thus, $2+1+8+6+5=22$; here again are *two Tens*, and two over to be set down under the second Column; then carrying the Two to the third Column of Hundreds, you say again, $2+6+3+7=18$; here is but *one Ten*, and 8 to be set down; then carrying one to the next Column, say $1+7+5=13$; here again is *one Ten*, and 3 to be set down; lastly, carry one to the last Place, and say $1+9=10$, which Number, be it what it will, is always set down, and the Sum total is complete in one Number, as in the Examples.

$$\begin{array}{r} 57 \\ 762 \\ 5389 \\ 97615 \\ \hline 103823 \end{array} \qquad \begin{array}{r} 6475 \\ 9830 \\ 2764 \\ 5937 \\ \hline 25006 \end{array}$$

26 The Reason why you carry *Ten* from every Column to the next, is because the Value of the Figures in each Column encreases in a ten-fold Proportion, as is evident from Article 9; and the Digits set down under each Column are *Units*, *Tens*, *Hundreds*, *Thousands*, *&c.* according to Inst. 11. which will express the Value of all the Columns severally collected and added together. Thus in the first Example, the Sums of each Column will stand thus, *viz.*

$$\text{The Sum}\left\{\begin{array}{rl} 23 & \text{Column of Units} \\ 200 & \text{——— of Tens} \\ 1600 & \text{——— of Hundreds} \\ 12000 & \text{——— of Thousands} \\ 90000 & \text{——— of Tens of Thousands.} \end{array}\right.$$

$$\overline{103823} = \text{Total Value.}$$

27. More

27. More Examples of Addition are the following:

5729	5009	847593
43605	120	215475
9834	369	176843
30546	1298	467890
75102	57306	150362
164816	64102	1858163

28. In the Addition of the ſeveral Sums of *Money*, *Meaſure*, *Weight*, *&c.* you obſerve one general Rule, *viz. to collect all the Units of one Species together; and then for every Number of thoſe Units, which make one of the next Species, you add one to the next Column, ſetting down the Remainder, as above.* In Money the Species are thus denoted,

viz.	*f.*	for	Farthings	of which	4 = 1 Penny.
	d.		Pence		12 = 1 Shilling.
	s.		Shillings		20 = 1 Pound.
	l.		Pounds		

Thus in the following Example, for every 4 Farthings you carry one Penny to the Column of Pence; for every 12 Pence you carry one to the Column of Shillings, and from thence, for every twenty, you carry one to the Place of Pounds; which are added as the Integers in Inſt. 27.

Example.

	l.	*s.*	*d.*	*f.*
	37	12	9	3
	125	7	10	2
	750	16	7	1
Total £.	913	17	3	2

29. I ſhall here ſubjoin ſeveral Sums to be added for the Learner's Practice.

l.	*s.*	*d.*	*f.*	*l.*	*s.*	*d.*	*f.*
175	19	6	2	3758	17	11	2
386	18	10	3	279	16	10	3
57	19	4	1	46	8	9	1
6	12	5	2	8	19	7	2

l.	s.	d.	f.
10	16	10	2
175	18	9	0
37	14	11	2
956	16	10	1
47	19	9	3

l.	s.	d.	f.
1000	16	10	2
100	10	11	3
90	2	3	2
10	12	10	1
9	17	5	1
0	17	4	0

30. Of *Weights* we have ſeveral Sorts, *viz.*

I. Troy Weight.

In which
- 24 Grains (*Grs.*) make 1 Penny Weight.
- 20 Penny Wts. (*Pwt.*) — 1 Ounce.
- 12 Ounces (*oz.*) —— 1 Pound. (*lb.*)

31.
II. Apothecaries Weight.

In which
- 20 Grains (*Grs.*) make 1 Scruple.
- 3 Scruples (℈) —— 1 Dram.
- 8 Drams (ʒ) —— 1 Ounce.
- 12 Ounces (℥) —— 1 Pound. (*lb.*)

32.
III. Averdupois Weight.

In which
- 16 Drams (*Dr.*) make 1 Ounce.
- 16 Ounces (*Oz.*) —— 1 Pound.
- 28 Pounds (*lb.*) —— 1 Quarter of an Hundred.
- 4 Quarters (*Qrs.*) — 1 Hundred Weight.
- 20 Hundred (*C.*) —— 1 Ton.

33. In each of theſe Species the Learner may exerciſe himſelf by the following Examples, *viz.*

In Troy Weight.

lb.	Oz.	Pwt.	Grs.
21	11	17	21
20	9	15	7
8	4	3	23
19	10	19	18

In

In Apothecaries Weight.

℔.	℥.	ʒ.	℈.	*Grs.*
18	10	7	2	16
9	11	5	0	10
16	9	4	1	10
3	5	6	2	15
25	9	0	1	6

In Avoirdupois Weight.

T.	*C.*	*Qrs.*	*lb.*	*Oz.*	*Drs.*
175	17	3	25	14	13
90	10	2	17	10	12
105	18	1	11	9	4
19	19	0	27	15	3
195	17	1	14	9	10
74	10	3	8	12	9

In Long Measure.

34. Our MEASURES are of divers Sorts; the Measures of Length are of the following Species.

3	Barley Corns make	1	Inch.
12	Inches — —	1	Foot.
3	Feet — —	1	Yard.
5½	Yards — —	1	Rod.
40	Rods — —	1	Furlong.
8	Furlongs — —	1	Mile.

35. Our CORN MEASURES are the following.

2	Gallons make	1	Peck.
4	Pecks ———	1	Bushel.
8	Bushels ——	1	Quarter.
5	Quarters —	1	Load.

36. Our LIQUID MEASURES are of the following Species, *viz.*

8	Pints make	1	Gallon.
9	Gallons —	1	Firkin of Beer.
8	Gallons —		Ditto of Ale.
2	Firkins —	1	Kilderkin.
2	Kilderkins	1	Barrel.
2	Barrels —	1	Hogshead.

37. But our WINE MEASURE is as follows, *viz.*

8	Pints make	1	Gallon.
63	Gallons	1	Hogshead.
2	Hogsheads	1	Pipe or Butt.
2	Butts —	1	Tun.

38. The true original Standard for Measures of Capacity or Solidity, is the CUBIC INCH.

Thus 282, 231, 268¾ Cubic Inches make 1 Ale Gallon, 1 Wine Gallon, 1 Corn Gallon.

Thus	Cubic Inches make	
282		1 Ale Gallon.
231		1 Wine Gallon.
268¾		1 Corn Gallon.

Also 1728 Cubic Inches make one Foot Solid.

 39. Our

39. Our Meaſures of TIME are of the following Denominations, *viz.*

60	Seconds make	1	Minute.
60	Minutes —	1	Hour.
24	Hours ——	1	Day.
7	Days ——	1	Week.
4	Weeks ——	1	Month.
13	Months ——	1	Year.

40. A CIRCLE, and alſo the MOTION performed in a Circle, (eſpecially in reſpect of the ECLIPTIC in aſtronomical Affairs) is divided in the following Manner, *viz.*

60	Seconds make	1	Minute.
60	Minutes —	1	Degree.
30	Degrees —	1	Sign.
12	Signs, or }	—	The Circle.
360	Degrees }		

In very nice Matters we ſub-divide a Second into 60 Thirds; a Third into 60 Fourths, and ſo on; as mentioned, Inſt. 15.

41. I ſhall ſubjoin the following Example for Practice.

Long Measure.

M.	*F.*	*Rds.*	*Yds.*	*F.*
175	5	36	3	2
19	3	27	5	1
76	7	39	4	0
121	0	20	2	1
54	6	17	4	0

M.	*F.*	*Rds.*	*Yds.*	*F.*	*In.*
195	3	27	3	1	10
79	7	30	4	0	11
9	6	29	5	2	9
17	5	20	4	0	7
59	4	18	0	1	6

Time.

Y.	*M.*	*W.*	*D.*	*H.*
175	9	3	2	17
140	12	1	5	21
35	7	0	6	19
9	5	2	4	11
110	10	1	3	8

Y.	*M.*	*W.*	*D.*	*H.*	′	″
143	10	2	5	19	50	37
74	9	3	6	20	7	57
3	4	0	3	1	39	10
19	3	1	2	5	25	41
158	12	2	0	22	30	8

Motion

Motion.

Rev.	Sg.	Dr.	′	″	Rev.	Sg.	Dr.	′	″
175	10	28	51	42	237	11	22	50	40
74	11	17	41	37	41	10	20	9	30
50	9	21	30	58	17	9	17	17	27
9	8	11	35	53	573	5	9	27	35
5	10	5	10	9	29	8	10	15	42
3	5	13	7	16	17	3	8	16	51

42. The beſt way to know if the Sum be caſt up right, is to do it twice over, beginning at the Bottom, and reckoning upwards the firſt Time; and then beginning at the Top, and reckoning downwards the ſecond Time; and if the *Sum Total* in both Caſes be the ſame, the Work is right, otherwiſe not.

CHAP. II.

Subtraction of Integers.

43. Subtraction is the ſecond Operation in the Art of *Computation*; it conſiſts in finding the Difference between two Numbers, *by taking the Leſſer from the Greater*; and as this is the Reverſe of Addition, ſo the Rule of performing it is the Contrary of that, *viz.*

RULE.

44. *From each Figure in the upper Line (beginning at the Right Hand) take the Figure in the under Line, if it be leſs, and ſet down the Difference, but if the Figure or Number in the lower Line exceeds that in the upper One which ſtands over it, encreaſe that upper Figure by adding* 10, *if it be in the Place of Integers; but if it be in any Species of Money, Weight, Meaſure,* &c. *you add to it ſuch a Number as makes One of the next Denomination, and then ſubtract as before, and ſet down the Difference, and remember, that every Time you thus encreaſe the upper Figure, you carry* one Unit *to the next Figure in the lower Line.*

45. This Rule will be best illustrated by the following Examples.

From	57238765	695278675
Take	13106243	434865740
Remains	44132522	260412935

5729438640	109765432016
178089305	16741320785

Money.

l.	*s.*	*d.*		*l.*	*s.*	*d.*	*f.*
175	17	9		17653	05	11	3
94	13	6		589	17	10	2
81	04	3		17063	08	01	1

l.	*s.*	*d.*	*f.*	*l.*	*s.*	*d.*	*f.*
9175	17	04	2	3729	17	10	0
1049	19	10	3	3018	17	11	3

Troy Weight.

l.	*Oz.*	*Pwt.*	*Grs.*	*l.*	*Oz.*	*Pwt.*	*Grs.*
131	10	18	14	173	05	00	01
79	11	19	20	9	10	00	23

Avoirdupois Weight.

T.	*C.*	*Qrs.*	*lb.*	*Oz.*	*T.*	*C.*	*Qrs.*	*lb.*	*Oz.*	*Drs.*
35	18	2	24	0	13	00	0	01	00	15
34	19	3	25	9	10	17	2	7	09	15

Long

Long Measure.

M.	*F.*	*Rds.*	*Y.*	*F.*
173	7	37	4	1
19	7	39	5	0

M.	*F.*	*Rds.*	*Y.*	*F.*	*In.*
517	6	00	1	2	10
9	0	00	4	2	11

Motion.

Rev.	*Sg.*	*Dr.*	′
15	07	20	40
14	11	29	50

Rev.	*Sg.*	*Dr.*	′	″
10	9	29	59	00
8	10	29	00	59

Time.

Y.	*M.*	*W.*	*D.*	*H.*
157	10	0	6	21
25	12	2	1	23

Y.	*M.*	*W.*	*D.*	*H.*	′	″
125	10	0	0	00	00	01
124	11	3	6	23	59	09

46. These Examples well underſtood, all others of every Sort will be eaſy; and the Reader needs not be told, that the Difference added to the lower Line, or leſſer Number, ought to make the Sum in the upper Line, and will, when the Work is right.

CHAP. III.

Multiplication of Integers.

47. Multiplication is the *third Operation* in Arithmetic, and is nothing more than a compendious Way of adding any Number or Sum, any Number of Times to its ſelf. Thus if the Number 135 be added 3 Times to its ſelf, the Sum will be the ſame as the Product of 135 multiplied by 3, as is evident below, *viz.*

Add

Add	135	Multiply	135
	135	By	3
	135	Product	405
Sum	405		

48. If the ſame Number 135 were to be made 13 Times as great, then you make it firſt, 3 Times as great, *viz.* 405; and after that, 10 Times as great (which is done by annexing a Cypher to the right Hand, Inſt. 9.) *viz.* 1350, and then the Sum of both theſe is that required, *viz.*

The Factors	135	The Multiplicand.
	13	The Multiplied.
1ſt Sum	405	
2d Sum	1350	
Total	1755	Product.

49. But that a Perſon may be ready at inveſtigating theſe particular Sums, it is abſolutely neceſſary to learn the following Table by heart.

The Multiplication Table.

3 Times	3 is	9	4 Times	4 is	16
	4	12		5	20
	5	15		6	24
	6	18		7	28
	7	21		8	32
	8	24		9	36
	9	27			
5 Times	5 is	25	6 Times	6 is	36
	6	30		7	42
	7	35		8	48
	8	40		9	54
	9	45			
7 Times	7 is	49	Times	8 is	64
	8	56		9	72
	9	63	9 Times	9	81

50. By

50. By this Table we find the Sum which ariſes by adding any Digit any Number of Times (under 10) to its ſelf; thus 8 added 7 Times to its ſelf makes 56, and therefore we ſay, 7 Times 8 is 56; and ſo of others. The Learner will take Notice, that 7 Times 5 is the ſame as 5 Times 7, and thereſore, it was not neceſſary to make the Table any larger.

51. Having this Table perfectly in Mind, the Operation will be eaſy by the following

RULE.

With the firſt Figure to the Right Hand, in the Multiplier, begin to multiply each Figure in the Multiplicand, proceeding from the Right to the Left; in the Product of each Figure, ſet down the Digits under 10, or between the Tens; and for every 10 contained in ſuch Product, carry One to the next Place. In like Manner proceed with every Figure of the Multiplier; and ſet down the ſeveral Products, one under another, in ſuch a Manner that they may ſtand one Figure backwards, or the firſt Figure of each Line under the ſecond Figure of that above; then add all into one Sum, which will be the Product, as in the following

Examples.

Mult.	1735	5738	4897
by	4	16	384
	6940	34428	19588
		5738	39176
			14691
		91808	
			1880448

52. The Reaſon of the laſt Part of the Rule, *viz.* ſetting the ſeveral Products one Figure back will eaſily appear from Inſt. 48. becauſe the Cyphers annexed to each Product are here rejected as making no Alteration in the Value of the Product, when thus reckoned up. Thus, in the laſt Example, the Multiplicand 4897, being multiplied by 4, 80, 300, ſucceſſively, makes the Products, 19588, 391760, 1469100, which, added together, make the ſame Number as before; as is evident below

```
   4897
    384
 -------
  19588
 391760   { Here the three Cyphers being rejected, as super-
1469100   { fluous, the Rest stands as before.
 -------
1880448
```

53. If a Cypher be found in the Multiplier, you place a Cypher under it, or annex it to the first Figure of the next Product; or you set the next Figure two Places back instead of one. Thus 1795 multiplied by 307 will stand

```
thus;   1795     or thus;  1795     or thus;  1795
         307                307                307
       -----               ----               ----
       12565              12565              12565
       0000              53850              5385
     5385                ------             ------
     -------             551065             551065
     551065
```

The last Way is most compendious, but the 2d is easiest for a Learner, and is generally used; the Reason of both appears from the first Position of the Product; for it is evident the 4 Figures 1795 multiplied by 0, gives 4 Places of Cyphers, which are of no Use to set down.

54. I shall here subjoin a Variety of Examples for the Learner's Observation and Exercise, both wrought and unwrought, as follow:

```
  1750        7580        80507
    76         520         3050
 ------      -------     ---------
  10500      151600       4025350
 12250      37900      2415210
 ------     -------    ----------
 133000     3941600     245546350
```

756940

756940	705059
10090	1000
68124600	705059000
75694000	
7637524600	
579603	1001000
10001	1010
579603	10010000
579603000	10010000
5796609603	1011010000
89704500	900203005
605730	90107004
57986000	100100010
18000500	10010010

55. The vulgar Method of proving the Truth of the Work, is by adding together the Digits in the Multiplicand and Multiplier, and casting out every *Nine*, as often as it occurs, noting the Remainders; then multiplying the Remainders, and casting the Nines out of the Product, you note the Remainder in that Case: Lastly, the Nines are cast out of the Sum of the Digits in the Product, and if the Remainder, in this Case, be the same with the last mentioned, it is a general Proof that the Work is true. Thus in the first Example of Inst. 54, the two first Remainders are 4 and 4; the Remainder in their Product is 7, which is also the Remainder in the whole Product of the Factors, and shews the Work is right. But the genuine Proof of Multiplication is by *Division*.

56. *Compound Multiplication* is sometimes of Use. This is when the *Multiplicand* consists of different Parts or Species of

Measure, *Weight*, *Time*, *Motion*, &c. As in the Examples following.

	l.	*s.*	*d.*	*lb.*	*oz.*	*pwts.*	*gr.*	*yds.*	*f.*	*in.*
Mult.	15	7	6 ½	8	11	16	10	40	2	10 ¼
By			5				8			12
	76	17	8 ½							

But when the Multiplier is large, these Operations are much easier by *Decimal Arithmetic*, as will be shewn farther on.

CHAP. IV.

Division of Integers.

57. By this Operation we find how often the smallest of any two given Numbers is contained in the greater, which is therefore but the Reverse of Multiplication; as for Example, Let it be required to find how often 3 is contained in 405. Then first, 3 is contained in 405, 100 Times, with a Remainder of 105; in this Remainder it is contained 30 Times with another Remainder of 15, in which it is contained 5 Times; therefore in all the Parts of 405, the Number 3 is contained, 100 + 30 + 5 = 135 Times, as in the Operation below. (See Inst. 47.)

Divisor.	*Dividend.*	*Quotient.*
3)	405	(100
	300	
3)	105	(30
	90	
3)	15	(5
	15	
	..	135

58. Hence you see the Reason of the compendious Form in the common Operation of this Part of Arithmetic; which therefore is performed by the following

RULE.

RULE.

If the Divisor be a less Number than so many Figures taken on the Left Hand in the Dividend make, see how often the Former is contained in the latter, and the Figure which expresses it, is the first of the Quotient; then multiply the Divisor by the Quotient Figure, and placing the Product under the said Figure or Figures of the Dividend, subtract it therefrom; and to the Remainder, annex the following Figure of the Dividend, which divide as before; and thus proceed 'till the whole Dividend be exhausted; as the Examples following.

```
3 ) 405 ( 135          8 ) 9769 ( 1221
    3                      8
    --                     --
    10                     17
     9                     16
    --                     --
     15                     16
     15                     16
    --                     --
     ..                     ..9
                              8
                             --
                              1

12 ) 1728 ( 144        135 ) 1755 ( 13
     12                      135
     ---                     ----
      52                      405
      48                      405
      --                      ---
       48                     ...
       48
       --
        .
```

59. If it happen that the Divisor be a greater Number than so many of the first Figures of the Dividend make, then you take a Number of Places in the Dividend greater by one, and proceed as before, as in the following Examples.

5) 3790 (758
35
—
29
25
—
40
40
—
..

25) 127900 (5116
125
—
29
25
—
40
25
—
150
150
—
...

60. If in any Case, the Remainder be so small, that when the Figure of the Dividend joined with it, make a Sum less than the Divisor, than a Cypher is to be placed in the Quotient, and another Figure taken down; and then the Division renewed, proceeds as above. Thus

9) 1863 (207
18
—
..63
63
—
..

12) 250836 (20903
24
—
108
108
—
...36
36
—
..

61. If the Divisor be not contained a whole Number of Times in the Dividend precisely, there will be a Remainder, when the Division is finished, which is to be placed Fraction-wise in the Quotient, together with the Divisor, as the Denominator, as in the following Cases.

7) 673 ($96\frac{1}{7}$
63
—
43
42
—
1

13) 12976 ($998\frac{2}{13}$
117
—
127
117
—
106
104
—
2

```
71 ) 29754 ( 419 5/71          131 ) 135076 ( 1031 15/131
     284                             131
     ---                             ---
      135                            ..407
       71                              393
      ---                              ---
      644                              146
      639                              131
      ---                              ---
      ..5                               15
```

62. When the Divisor has one or more Cyphers at the Beginning, they may be omitted in the Work, provided you strike off as many Figures in the Beginning of the Dividend, which are to be annexed to the Remainder to compleat it; as will appear in the ensuing Operations.

```
*At Length.*                     *Contracted.*
70 ) 5689 ( 81 19/70             7|0 ) 568|9 ( 81 19/70
     560                               56
     ---                               --
      89                               ..8
      70                                 7
      --                                --
      19                                19

8300 ) 9850716 ( 1186            83|00 ) 98507|16 ( 1186
       8300                              83
       ----                              --
       15507                             155
        8300                              83
       -----                             ---
       72071                             720
       66400                             664
       -----                             ---
        56716                             567
        49800                             498
        -----                             ---
        .6916                             6916
```

63. These are all the Varieties the Learner will meet with in this Operation; and the following Examples will exercise him therein.

175) 89463 (9850) 82768 (
5900) 300012 (15700) 9875830 (
1000) 980005 (60800) 908437200 (

64. To prove the Truth of the Work, multiply the Quotient by the Divisor, and if the Product be the same with the Dividend, the Work is right, otherwise not. Note, if there be any Remainder in the Division, it must be added to the said Product to make the Proof. Thus in the last Example of (61) we have $1031 \times 131 = 135061$, and then the Remainder 15 being added, the Whole makes 135076, the same with the Dividend.

65. And, *vice versâ*, the only true Proof of Multiplication is by Division, as will easily appear to any one who understands the preceding Operations. Thus in the first Example of (54) if you divide the Product 133000 by the Multiplier 76, the Quotient will be the Multiplicand 1750, which proves the Work true.

CHAP. V.

Of Vulgar Fractions.

66. TO fit Vulgar Fractions for Operation, they should be first of all reduced to their lowest Denomination, which is done by finding a common Divisor by this

RULE.

Divide the Denominator by the Numerator, and the Numerator by the Remainder (if any) and that first Remainder by the Second, the second by the Third, and so on, till the Remainder be nothing; as in the Instances following, viz. $\frac{15}{815}$, $\frac{31}{157}$.

```
15 ) 81 ( 5                31 ) 157 ( 5
     75                         155
     --                         ---
      6 ) 15 ( 2                  2 ) 31 ( 15
          12                           30
          --                           --
           3 ) 6 ( 2                    1 ) 2 ( 2
               6                            2
               -                            -
               .                            .
```

67. The laſt Diviſor is the *common Diviſor* ſought, by which, if the Fraction be divided, it will be reduced to its loweſt Terms; thus if the Fraction $\frac{15}{81}$ be divided in each Part by 3, it is reduced to $\frac{5}{27}$, which is equal in Value to the former. And becauſe, in the ſecond Example, the Remainder is Unity, it ſhews that Fraction is already in its loweſt Denomination.

68. *Mix'd Numbers* muſt be reduced to a fractional Form, which is done by this

RULE.

Multiply the integral Part by the Denominator of the fractional Part, to which add the Numerator, and the Sum will be a new Numerator; under which write the Denominator, and it makes an improper Fraction of the ſame Value. As in theſe Examples, $5\frac{2}{7}$, $1\frac{3}{11}$, $28\frac{1}{13}$. *Thus* $5 \times 7 = 35$, *and* $35 + 2 = 37$, *then* $\frac{37}{7} = 5\frac{2}{7}$; *alſo* $1 \times 11 = 11$, *and* $11 + 3 = 14$, *then* $\frac{14}{11} = 1\frac{3}{11}$; *laſtly*, $28 \times 13 = 364$, *and* $364 + 1 = 365$, *then* $\frac{365}{13} = 28\frac{1}{13}$. *The Reaſon of all which appears from common Diviſion* (61.)

69. Two or more Fractions are reducible to *one common Denomination*, that is, ſhall have all the ſame Denominator, retaining ſtill their firſt Values, by this general

RULE.

Multiply the Numerator of each Fraction into all the Denominators, but its own, for a new Numerator, and multiplying all the Denominators together, the Product ſhall be a common Denominator to all the Numerators before found.

Thus let it be required to reduce $\frac{2}{5}$, $\frac{7}{13}$, $\frac{17}{73}$, to one common Denomination, the Work will ſtand as below.

2

2	7	17	73
13	5	13	13
26	35	51	219
73	73	17	73
78	105	221	949
182	245	5	5
1898	2555	1105	4745

Then the Fractions become $\frac{1898}{4745}$, $\frac{2555}{4745}$, $\frac{1105}{4745}$, all of the same Value as before.

70. This previous Reduction of Fractions to a common Denominator, renders them easy to be added, or subtracted; for this is done among the Numerators only of the Fractions thus reduced; for Example, to add the two Fractions $\frac{2}{5}$ and $\frac{7}{13}$ together; these reduced (*per* 69) are $\frac{1898}{4745}$ and $\frac{2555}{4745}$, whence $1898 + 2555 = 4453$; consequently $\frac{4453}{4745} = \frac{2}{5} + \frac{7}{13}$, the Sum required. Thus also, $\frac{2}{5} + \frac{7}{13} + \frac{17}{73} = \frac{5558}{4745} = 1\frac{813}{4745}$. So any other Fractions are added together.

71. In like Manner you subtract one Fraction from another; thus $\frac{7}{13} - \frac{2}{5} = \frac{657}{4745}$, because $2555 - 1898 = 657$. So also $\frac{2}{5} - \frac{17}{73} = \frac{1898-1105}{4745} = \frac{793}{4745}$. And the like of others. When Fractions are thus added or subtracted, the fractional Sum or Difference is to be reduced to its lowest Denominator by the Rule of (66.)

72. To *multiply Fractions* together, is no more than to multiply the Numerators and the Denominators among themselves, and the Product thence arising will be that which is required. Thus $\frac{2}{5} \times \frac{7}{13} = \frac{14}{65}$. Thus also $\frac{7}{13} \times \frac{17}{73} = \frac{119}{949}$; so $\frac{1}{9} \times \frac{7}{5} = \frac{7}{45}$.

73. To *divide one Fraction* by another, you multiply the Denominator of the Divisor by the Numerator of the Dividend, for the Numerator of the Quotient; then you multiply the Numerator of the Divisor by the Denominator of the Dividend for the Denominator of the Quotient. Thus $\frac{14}{65}$ divided by $\frac{2}{5}$ will stand thus, $\frac{2}{5})\frac{14}{65}(= \frac{70}{130} = \frac{7}{13}$; and thus $\frac{7}{13})\frac{119}{949}(\frac{1547}{6643} = \frac{17}{73}$, by Reduction. Lastly, $\frac{1}{9})\frac{7}{45}(\frac{63}{45} = \frac{7}{5} = 1\frac{2}{5}$. These Cases of Division being only the Reverse of those of Multiplication (72.)

74.

47. Any whole Number is expressed Fraction-wise, by only writing Unity under it, thus 5 is $\frac{5}{1}$, 17 is $\frac{17}{1}$, &c. And so Integers and Fractions may be added, subtracted, multiplied or divided in the same Manner, as pure Fractions themselves by the preceding Rules. Thus $\frac{2}{15} \times 5 = \frac{2}{15} \times \frac{5}{1} = \frac{10}{15} = \frac{2}{3}$. On the contrary, $5)\frac{2}{3}(= \frac{5}{1})\frac{2}{3}(= \frac{2}{15}$.

CHAP. VI.

The Reduction of Quantities of divers Denominations into One.

75. THE Rule for doing this is to multiply each superior Species by the Number which it contains of the proposed Species below it. Thus one Pound is reduced to Numbers expressing an equal Value in all the inferior Species; as also other Quantities, as in the Examples below.

1 *l. Sterling.*
20
—
20 *Shillings.*
12
—
240 *Pence.*
4
—
960 *Farthings.*

1 *lb. Troy.*
12
—
12 *Ounces.*
8
—
96 *Drams.*
3
—
288 *Scruples.*
20
—
5760 *Grains.*

1 *lb. Avoirdupois.*
16
—
16 *Ounces.*
16
—
256 *Drams.*

1 *Tun.*
20
—
20 *Hundreds.*
4
—
80 *Quarters.*
28
—
2240 *Pounds.*
16
—
35840 *Ounces.*

1 *lb. Apothecaries Wt.*
12
———
12 *Ounces.*
20
———
240 *Penny Wt.*
24
———
5760 *Grains.*

1 *Mile*
8
———
8 *Furlongs.*
40
———
320 *Rods.*
$16\frac{1}{2}$
———
5280 *Feet.*
12
———
63360 *Inches.*

1 *Day.*
24
———
24 *Hours.*
60
———
1440 *Minutes.*
60
———
86400 *Seconds.*

1 *Circle.*
12
———
12 *Signs.*
30
———
360 *Degrees.*
60
———
21600 *Minutes.*
60
———
1296000 *Seconds.*

76. If the Quantity, to be reduced, consists of different Species, the Number of each several Species is to be taken in, or added to the Product which is of the same Kind or Denomination, as you proceed in the Reduction; according to what you see here exemplified.

l.	*s.*	*d.*	*f.*
15	17	4	3
20			
317 *s.*			
12			
3808 *p.*			
4			
15235 *f.*			

lb.	*oz.*	*pwts.*	*grs.*
9	7	15	21
12			
115			
20			
2315			
24			
9261			
4632			
55581			

Hence

Hence several Questions may be proposed for the Learner's Exercise, as follow.

	lb.	*oz.*	*drs.*			
In	35	13	15	Avoirdupoise Weight, how many Drams?		
	T.	*C.*	*Qrs.*	*lb.*		
In	17	13	3	25, how many Pounds?		
	M.	*F.*	*R.*	*F.*		
In	57	7	37	15, how many Feet?		
	D.	*H.*	′	″		
In	275	17	39	48, how many Seconds?		
	R.	*S.*	°	′	″	
In	15	10	27	49	53, how many Seconds?	
	Lds.	*Qrs.*	*B.*	*G.*		
In	35	4	7	5, how many Gallons?		
	Hgs.	*Gal.*	*Pts.*			
In	13	51	7, how many Pints?			

77. When we have any large Number expressing any Quantity in its lower Species, the same may be reduced to any or all the higher Species by Division, being only the Reverse of the other Process by Multiplication ( ).

Thus suppose it were required to know how many Pounds, Shillings, Pence and Farthings were contained in 15235 Farthings? 'Tis evident, *if we divide first by* 4, *the Quotient will be Pence; this divided by* 12, *will quote Shillings; and these again divided by* 20, *will shew the Number of Pounds; the Remainders in each Division being the odd Shillings, Pence, and Farthings.* The Work placed after the usual Manner will stand as below.

```
               12       20
   4 ) 15235 ( 3808 ( 317 ( 15
       12      36     20
       --      --     --
        32      20    117
        32      12    100
        --      --    ---
       ..35     88     17
         32     84          Answer 15l. 17s. 4d. 3f.
         --     --
         .3     .4
```


Again,

Again, In 55581 Grains, how many Pounds, Ounces, Pennyweights, and Grains?

```
              20        12
24 ) 55581 ( 2315 ( 115 ( 9
     48        20       108
     —         —        ———
     ·75       31       ·7
      72       20
      —        —
       38      115
       24      100
       —       ———
       141     15
       120
       ———
        21
```

	lb.	*oz.*	*pwt.*	*grs.*
Answer	9	7	15	21

After the same Manner the Learner may operate the following Questions.

In 59786 Grains, how many Pounds, Ounces, Drams, Scruples, and Grains? Troy.

In 197568 Ounces, Avoirdupois, how many Tuns?

In 1000000 Feet, how many Miles, *&c.*

In 2974600″, how many Days?

In 5973864 Pints, how many Load?

CHAP. VII.

Of Decimal Arithmetic.

78. THE Nature and Notation of Decimal Numbers having been already declared (12, 13, 14,) I shall proceed immediately to give the Rules for their Operation. The first of which is

ADDITION.

The *Addition* of Decimal Numbers does no way differ from that of Integers, due Care being taken to place all the particular Sums, so that the first Places of the integral or decimal Parts be exactly under each other, as is seen in the following Examples. (See 25, 27.)

5,7	64,75	5,729
76,2	98,30	43,605
538,9	27,64	9,834
9761,5	59,37	30,546
		75,102
10382,3	250,06	164,816

79. If any Number be purely decimal, or has no integral Part, 'tis usual to put a Cypher in Units Place of Integers, as thus,

0,057	0,5729
0,762	4,3605
5,389	0,9834
97,615	3,0546
	7,5102
103,823	16,4816

80. Cyphers on the Right Hand of a decimal Number avail nothing, and are therefore more elegantly omitted.

Thus instead of	we write
0,0100	0,01
53,9270	53,927
4,1009	4,1009
975,2300	975,23
87,1200	87,12
1120,3879	1120,3879

SUBTRACTION.

81. The same Precautions being observed, *Subtraction* of Decimals is performed in all Respects like that of Integers, as in the Examples following. See (44.)

5723,8765	69527,8675
1310,6243	43486,574
4413,2522	26041,2935

572,0987

572,0987	1435,2
179,5	1379,8256
392,5987	0055,3744
0,5697631	5.
0,5589030	4,876902
0,0108601	0,123098
10,005	100000,
0,0098753	9,0000001
9,9951247	99990,9999999

MULTIPLICATION.

82. In *Multiplication* of Decimals, having placed the Factors (as in common Multiplication, 51.) you observe this general Rule, *viz. Cut off so many Places of Figures for Decimals in the Product as there contain'd Decimal Places in both the Factors.*

Thus Mult.	17,35	57,38	48,97
by	4	1,6	3,84
	69,40	34428	19588
		5738	39176
		91,808	14691
			188,0448

1750	0,758	0,80507
0,76	0,52	0,305
10500	1516	402535
12250	3790	2415210
1330.	0,39416	0,24554635

83. But if it happens that when the Operation is finished, there are not so many Figures in the Product, as there are Places of Decimals in the Factors, then Cyphers are to be prefix'd to the

the Product to make the Number of Places equal. As in these Examples.

,3	,09	,00175	,001
,2	,08	,0076	,01
,06	,0072	1050	,00001
		1225	
		,000013300	

59.	,1000101	1000000.
,000006	,1	,00000001
,000354	,01000101	,01

DIVISION.

84. Division of Decimals is also perform'd by one general Rule, *viz. Divide the Numbers as if they were all Integers, and then cut off so many Figures for Decimals in the Quotient as when added to those of the Divisor, do make the same Number of Decimals as are contain'd in the Dividend.* Examples follow. (See 51, 54.)

1,6) 91,808 (57,38
80
11 8
11 2
60
48
128
128

48,97) 188,0448 (3,84
146 91
41 134
39 176
1 9588
1 9588

1750) 1330,00 (,76
1225 0
105 00
105 00

,52) ,39416 (,758
364
301
260
416
416

85. I

85. If, when the Division is finished, there are not so many Places in the Quotient as with those Decimals in the Divisor will equal the Number of Decimals in the Dividend, then Cyphers must be prefixed to equal that Number. As in these Examples.

```
,09 ),0072( ,08        59 ),000354 ( ,000006
      72                     354
      --                     ---
      ..                     ...
```

86. When the Dividend is an integral Number, so many Cyphers are to be annexed thereto as there are decimal Places in the Divisor; and the Quotient is in this Case integral.

```
Thus 2,5 ) 425.0 ( 170.        ,39 ) 741.00 ( 1900.
           25                        39
           ---                       ---
           175                       351
           175                       351
           ---                       ---
           ...0                      ...00

5,116 ) 1279.000 ( 250.        ,001 ) 597,000 ( 597000.
        1023 2                        597
        ------                        ---
         255 80                       ...
         255 80
         ------
             .0
```

87. But if, when the Division is thus far finished, there happens to be any Remainder; then more Cyphers are to be annexed to the former in the Dividend, and so many Places are Decimal in the Quotient, as is evident in the following Examples. (See 61.)

```
,7 ) 673.0|00 ( 961,42
     63
     --
      43
      42
      --
       10
        7
       --
        30
        28
        --
         20
         14
         --
          6
```

```
1,3 ) 12976.0|000 ( 9981,538
      117
      ---
       127
       117
       ---
        106
        104
        ---
          20
          13
          --
           70
           65
           --
            50
            39
            --
            110
            104
            ---
              6
```

```
,71 ) 29754,00|000 ( 41907,042
      284
      ---
       135
        71
       ---
        644
        639
        ---
         500
         497
         ---
          300
          284
          ---
           160
           142
           ---
            18
```

```
,015 ) 1.000|00 ( 66,66
         90
         --
         100
          90
         ---
         100
          90
         ---
         100
          90
         ---
          10
```

88. N. B. In Case any large decimal Numbers are to be multiplied by each other, and it be required to retain only a certain Number of decimal Places in the Product, you may contract the Work and shorten the Labour by proceeding as follows, viz. *Write down the Multiplicand as usual, and then write under it the Multiplier inverted, with the Unit's Place thereof under that Place of the Multiplicand, whose Place you intend the Product shall extend to; then multiply, as usual, by each Figure of the Multiplier, beginning*

ginning at those of the Multiplicand which stand over it, neglecting those to the Left, unless so far as to observe what would arise from multiplying the Figure immediately foregoing, which must be taken in at the Beginning of each Line, the first Figure of all which must stand under one another on the Right-hand.

89. Thus let it be required to multiply 3,141592 by 52,7438 to have four Places of Decimals in the Product. And also 104226,8672 by ,261799388; the Work for each will stand as below.

Multiplicand	3,1415 92	104226,867 2
Multiplier inverted	8347,25	883997 162,0
	1 5707 96	20845 373 4
	628 32	6253 612 0
	219 91	104 226 8
	12 57	72 958 7
	94	9 380 3
	25	937 9
		31 2
Product =	165,69 95	8 3
		8
		Product = 27286,529 4

90. The Reason of this Contraction will easily occur to any one who considers the Work at large, as it stands below for the first of these Examples.

3,1 4	1592
5 2,	7438
2 5	1 32736
9 4	2 4776
125 6	6 368
2 199 1	1 44
6 283 1	8 4
157 079 6	0
165,699 5	0 01296

The perpendicular Line here drawn among the Figures cuts off all the superfluous Part of the Work to the Right, and leaves the significant Part on the Left, which is the same as the contracted Part, but in an inverted Order, which is the Reason why the Multiplier is inverted in that Case to produce it.

N. B. What relates to the Doctrine of *Repetends* or *circulating Decimals*, we shall refer to *Logarithms*; as this intricate Affair is most easily manageable by those *artificial Numbers*.

CHAP.

CHAP. VIII.

The Reduction of Vulgar Fractions, and Quantities of divers Species to Decimal Numbers.

91. A Common Fraction is reduced into a decimal Expression of the same Value by dividing the Numerator by the Denominator by the Rules in the preceding Chapter. As in the Examples below.

```
2/5 = 5 ) 2,0 ( = 0,4
          2 0
          ---
          . .
```

```
3/8 = 8 ) 3,000 ( = 0,375
          24
          --
           60
           56
           --
            40
            40
            --
            . .
```

```
1/6 = 6 ) 1,000 ( = 0,16
          6
          -
           40
           36
           --
           .4
```

```
5 3/8 = 43/8 = 8 ) 43,000 ( 5,375
                   40
                   --
                    30
                    24
                    --
                     60
                     56
                     --
                      40
                      40
                      --
                      . .
```

92. By this Means you find the decimal Value of any Species of *Money*, *Weight*, *Measure*, *Time*, *Motion*, &c. in any Denomination above it. *viz.* By dividing Unity by the Number expressing how many of that Species make One in the superior

Denomination proposed. Thus because four Farthings make one Penny; therefore $\frac{1}{4} = 0{,}25$ is the Decimal of a Penny for one Farthing. In like Manner, $\frac{1}{12} = 0{,}083 =$ the decimal Part of one Shilling for a Penny; and $\frac{1}{20} = 0{,}05 =$ the Decimal of a Pound for one Shilling.

93. Again, if it be required to know what decimal Part of a Pound one Farthing is, because 960 Fathings make a Pound, therefore $\frac{1}{960} = 0{,}0010416$ is the Decimal of a Pound for one Farthing. After this Manner the following Tables are made, shewing the decimal Value for each Species of the integral Quantity or highest Denomination, which is always Unity, or 1, in the several Sorts of Money, Weight, Measure, &c.

One Pound	1.	*Troy Pound*	1.
Shilling	0,05	*Ounce*	0,083
Penny	0,00416	*Dram*	0,010416
Farthing	0,0010416	*Scruple*	0,003472
		Grain	0,0001736
One Tun	1.	*One Mile*	1.
Hundred	0,05	*Furlong*	0,125
Quarter	0,0125	*Rod*	0,003125
Pound	0,000446	*Foot*	0,000189
Ounce	0,000028	*Inch*	0,000016
One Day	1.	*A Circle*	1.
Hour	0,0416	*Sign*	0,083
Minute	0,000694	*Degree*	0,0027
Second	0,0000011	*Minute*	0,00004
		Second	0,0000006
One Load	1.	*One Tun of Wine*	1.
Quarter	0,2		
Bushel	0,025	*Pipe*	0,5
Gallon	0,003125	*Hogshead*	0,25
Pint	0,00039	*Tierce*	0,16
		Gallon	0,003967.

94. Having any Quantity expressed in divers Species, as 15*l.* 17*s.* 4*d.* 3*f.* 'tis easy, by the Tables, to turn it into Decimals,

Thus

$$\text{Thus}\begin{cases}1 \times 15 = 15\,l.\\ 0,05 \times 17 = 0,85\\ 0,00416 \times 4 = 0,016\\ 0,001 \times 3 = 0,003\end{cases}$$

Therefore 15*l.* 17*s.* 4*d.* 3*f.* = £ 15,869

95. Hence, by this Method, it will be eaſy to perform ſome Operations in Arithmetic, which would otherwiſe prove very irkſome and difficult. Thus, ſuppoſe it was required to

	Feet	*Inches*		
Multiply	12	: $9\frac{3}{4}$	=	12,8125
By	7	: $5\frac{1}{2}$	=	7,4582

Now this would greatly puzzle a Learner to perform in the common Way, but when reduced to Decimals, is only a Caſe of common Multiplication. And it ſhould be a Maxim with Schoolmaſters never to torture the Genius of a Scholar with things extremely difficult, and at the ſame Time unneceſſary, or performable by eaſier Methods; which is but too commonly the Caſe.

CHAP. IX.

Of the Ratios of Numbers, and the Rules of Proportion, or Rule of Three Direct and Inverſe.

96. AS the Ratio of Numbers conſiſts of a Compariſon or Relation in reſpect of Magnitude; and as one Number may exceed another both by *Addition* and *Multiplication*, therefore two Sorts of Ratios will ariſe in the Compariſon of Numbers, *viz.* one, which will be expreſſed by their Difference when the leſſer is taken from the greater; and another, which will be expreſſed by the Quotient, in dividing the greater by the leſſer.

97. Hence, in this Series of Numbers 1, 2, 3, 4, 5, 6, 7, 8, &c. where each is greater or leſs than the other by Addition or Subtraction of 1, the common Difference will be 1 between any two contiguous Terms, which is therefore called the *common*

Ratio

Ratio of the Series; thus the Ratio of 2 to 1, is $2 - 1 = 1$; of 5 to 4, is $5 - 4 = 1$; and so of the Rest. But the Ratio of 3 to 1, is $3 - 1 = 2$, and 5 to $3 = 5 - 3 = 2$, which is double the former. And the Ratio of 4 to $1 = 4 - 1 = 3$, and 5 to $2 = 5 - 2 = 3$, &c. which are triple the First. In the following Series 1, 3, 5, 7, 9, 11, 13, 15, &c. the Ratio or common Difference is 2. In this Series 5, 9, 13, 17, 21, 25, &c. the Ratio is 4; and so it may be any given Number, by which any Series of Numbers encrease or decrease. And hence such a Series of Numbers are said to be in *arithmetical Progression.*

98. The other Sort of Ratio is that between Numbers which differ by a *common Multiplication or Division*; Thus, if 1 and each subsequent Product be constantly multiplied by 2 this Series will arise, *viz.* 1, 2, 4, 8, 16, 32, 64, &c. and this constant Multiplier is the *common Ratio of the Series*; for the Ratio of any two proximate Numbers is the same; thus the Ratio of $2 : 1$ is $\frac{2}{1} = 2$; and the Ratio of $16 : 8 = \frac{16}{8} = 2$. Again, in this Series 1. 3. 9. 27, 81, 243, &c. the Ratio is 3; thus $3 : 1 = \frac{3}{1} = 3$, or $243 : 81 = \frac{243}{81} = 3$, and so of others.

99. In these Series, the Ratio of the third Term to the first is not *double* that of the Second to the First, (as in arithmetical Series) but is said to be *duplicate of it*; thus $4 : 1 = 2 \times 2 = 4$, whereas $2 : 1 = 2$, in the first Series; and in the second Series, $9 : 1 = 3 \times 3 = 9$, whereas $3 : 1 = 3$. Again, the Ratio of the fourth Term to the First is not *triple* (as in 91) but *triplicate* of the Ratio of the second to the first. Thus, $8 : 1 = 2 \times 2 \times 2 = 8$, which is triplicate of $2 : 1 = 2$; or this latter Ratio is three Times involved in the former. Thus also $27 : 1 = 3 \times 3 \times 3 = 27$, which therefore is *triplicate* the Ratio of $3 : 1 = 3$. Whence a Series of Numbers having such a Ratio is said to be in *geometrical Progression,*

100. Hence it is evident, that in order to make any Ratio twice as great as before, it must be multiplied by itself. Thus $2 : 1$ added to $2 : 1$, is $\frac{2}{1} \times \frac{2}{1} = \frac{4}{1}$; or $\frac{3}{1} \times \frac{3}{1} = \frac{9}{1}$, is $3 : 1$ added to it self, or made twice as great. Consequently, the *Addition of geometrical Ratios* is performed by multiplying those Ratios by each other. Thus the Sum of the Ratios $5 : 3$ and $8 : 7$ is $\frac{5}{3} \times \frac{8}{7}$

=

$=\frac{40}{21}=40:21$. Again, $1:3$ added to $5:9$ makes the Sum equal to the Ratio $5:27$, becaufe $\frac{1}{3}\times\frac{5}{9}=\frac{5}{27}$.

101. On the other Hand, the *Subtraction of the geometrical Ratios* is performed by dividing the greater by the leffer. Thus, if from $4:1$ I take $2:1$, there will remain $2:1$, becaufe $\frac{2}{1}$) $\frac{4}{1}$ ($=\frac{2}{1}$. In like Manner, if from the Ratio $40:21$ I fubduct the Ratio of $8:7$, there will remain the Ratio of $5:3$ for $\frac{8}{7}$) $\frac{40}{21}$ ($=\frac{5}{3}$. And from the Ratio $5:27$ if we take $5:9$, there will remain $1:3$; becaufe $\frac{5}{9}$) $\frac{5}{27}$ ($=\frac{1}{3}$.

102. From a Comparifon of geometrical Ratios refults the *Doctrine of Proportion or Analogy* (21) for if there be three Numbers, fuch that the Ratio between the firft and fecond be the fame with the Ratio between the fecond and third, then are thofe Numbers faid to be *Proportionals*, as 1, 2, 4; for $1:2=2:4$, or $1:2::2:4$. Alfo 1, 3, 9, are *proportional*, becaufe $1:3=3:9$ (by 98.) Thus alfo four Numbers are proportional, when the Ratio is the fame between the firft and fecond, as it is between the third and fourth; as 1, 2, 4, 8, or 1, 3, 9, 27; becaufe $1:2=4:8$; and $1:3=9:27$.

103. Thefe proportional Numbers are fuch as either fucceed each other immediately in the Series, as $1:2::4:8$, or $4:8::16:32$; and the Proportion is faid to be *continued*, and fuch Terms are called *continual Proportionals*. But if the Ratios are taken between fuch Pairs of Numbers as do not ftand together, or immediately follow each other in the Series, then is the Proportion faid to be *difcontinued* or *disjunct*. As in thefe $1:2::8:16$, or $4:8::32:64$; or $1:3::81:243$. And this makes what is vulgarly called the *Golden Rule* (becaufe of its Ufefulnefs) or *Rule of Three*.

104. For by this Rule, if any three Numbers are given as 3, 9, 81, a Fourth may be found which fhall be in Proportion, that is, fhall have the fame Ratio to the third Term 81, as there is between the two firft 3 and 9. And becaufe this fourth Term is as the unknown or fought, let us call it x. Then by Suppofition, $3:9::81:x$; therefore $\frac{9}{3}=\frac{x}{81}$ (by the Nature of the Series 98.)

105. Now it is an *Axiom, that if equal Things are multiplied by equal Things, the Products will be equal*; therefore if the two equal Ratios $\frac{9}{3}$, $\frac{x}{81}$ be each multiplied by the same Number 81, we shall have $\frac{9 \times 81}{3} = \frac{x \times 81}{81} = x$, as is evident because $\frac{81}{81} = 1$, and so makes no Alteration in the Value of x; therefore the Rule is, *multiply the second and third Numbers together, and divide the Product by the First, the Quotient will be the fourth Number sought.* Thus $\frac{9 \times 81}{3} = 243$, so that $3 : 9 :: 81 : 243$, according to (98.)

106. Hence this Rule comes to be of very great and frequent Use in the various practical Affairs of Life, which I shall exemplify by a few Questions, as follows.

If 3 Yards cost 9 Shillings, what will 81 Yards cost? See the Operation.

```
Yds.  S.  Yds.    £.   S.
  3 : 9 :: 81  :  12   3
            9
          ———     20
     3 ) 729  (  243 ( 12l.
          6       20
          —       ——
          12      43
          12      40
          ——      ——
          ..9     .3
            9              Answer 12l. 3s.
           ——
            .
```

If 100*l.* gain 5*l.* Interest, what will 750*l.* gain in the same Time?

```
 l.    l.     l.     l.   s.
100 :  5  :: 750  :  37   10
              5
          ————————
  100 ) 3750,0 ( 37,5 = 37l. 10s.
        3750 0
        ——————
        .....
```

If

If the Moon describes the whole Ecliptic, or 360° in $27\frac{1}{2}$ Days, how many Degrees does she pass thro' in one Day?

$$\begin{array}{lcccc} & D. & & & D. \\ \textit{Analogy.} & 27{,}5 : 360° & :: & 1 : & 13°{,}09 \end{array}$$

```
           1
         -----
27,5 ) 360 ( 13,09, Answer.
       275
       ----
        850
        825
        -----
         2500
         2475
         ----
          ..35
```

According to the accurate Measures of the *French*, there are 57060 *Toises* in a Degree, or 242360 *Paris* Feet; the Circumference therefore of a great Circle is 123249600 Feet; and the *Paris* Foot is to the *English*, as 1068 to 1000: Quere, how many *English* Feet and Miles are in the Earth's Circumference?

		Paris Feet.	*Eng. Feet.*		
Analogy 1000 : 1068 ::		123249600 :	131630573		

Then because 5280 Feet = 1 Mile, say:

F.	*M.*	*F.*	*M.*	*Yds.*	*F.*
As 5280 :	1 ::	131630573 :	24930	57	2.

107. When the *Ratio* of the Series is carried on by Division below the first Term, as it is above it by Multiplication, as thus, $16 : 8 : 4 : 2 : 1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16}$, &c. then are the *Ratios* of these fractional Numbers to Unity, said to be *inversly*, or *reciprocally* as the *Ratios* of the integral Numbers to Unity; that is, the *Ratio* of $\frac{1}{8}$ to 1 is the Reciprocal of 8 to 1; or $\frac{1}{8}$ is as *much less* as 8 is *greater* than Unity, or 1.

108. When, therefore, any Question in the Rule of Three is proposed, and is of such a Nature, that the fourth Number x, or Consequent of the second Ratio of the Analogy is reciprocally to its Antecedent of what the Consequent in the first Ratio is to its Antecedent, then the *Rule of Three* is said to be *Inverse*.

And the Terms muſt be ſtated in a contrary or inverſe Order, as in the following Queſtion.

If 12 *Men do a Piece of Work in* 15 *Days, in how many Days will* 20 *Men do the ſame?*

Here it is evident, the 4th Number x cannot ſtand in the 4th Place, as before, (104 *viz.* $12 : 15 :: 20 : x$; for then x would be directly to 20, as 15 to 12; but x muſt be leſs than 20, and therefore reciprocally as 15 to 12. Again, it is plain, that $\overset{M.}{20} : \overset{M.}{12} :: \overset{D.}{15} : x$ Days; and becauſe the Proportion is here direct, therefore $\frac{12 \times 15}{20} = x$ (105.) Conſequently, if we take the Terms as they ſtand in Queſtions of this Sort, the Rule for operating them in this, *Multiply the firſt and ſecond Numbers together, and divide by the Third, and the Quotient will be the fourth Number* (x) *ſought.* So in the preſent Caſe $\frac{12 \times 15}{20} = x = 9$ Days, the Anſwer.

109. I ſhall here ſubjoin a Queſtion or two of this Sort, as follows.

A Friend lends me 372l. *for* 7 *Years and* 8 *Months, how long muſt I lend him* 496l. *for an Equivalent?*

$\frac{372 \times 7,6}{496} = 5,7$ Then per Rule $\frac{496 \times 7,6}{372} = 5,75$ Years, Anſwer.

If 3 *Men, and* 4 *Women, can do a Piece of Work in* 56 *Days, how long will one Man and one Woman be doing the ſame?*

Becauſe of 3 Men and 4 Women, ſome Number muſt be found that may be divided by 3 and 4 without a Remainder, as the Number 12; then make the 3 Men or 4 Women equal to 12 Boys; and 3 Men and 4 Women will be equal to 24 Boys; and 1 Man will be equal to 4 Boys, and 1 Woman to 3 Boys, and 1 Man and 1 Woman to 7 Boys; then the Queſtion is reduced to this, *If* 24 *Boys do a Piece of Work in* 56 *Days, in how many Days will* 7 *Boys do the ſame?*

$$\text{Anſwer } \frac{24 \times 56}{7} = 192 \text{ Days.}$$

110. I ſhall here ſay nothing of the *compound Rule of Proportion*, or, as it is uſually called, the *Double Rule of Three*, wherein five Numbers are given to find a ſixth, by means of two Analogies,

gies, becauſe this will be beſt explained, and the Reaſon of Operation will be more evidently ſeen, in the Method of treating this Subject in Algebra.

111. From what has been ſaid of Ratios, it is eaſy to make a Compariſon of their Magnitude, having firſt premiſed this Definition, *viz. That Ratio is ſaid to be* greater, equal to, or leſs *than another, whoſe Antecedent hath* a greater, or an equal, or a leſs *Proportion to its Conſequent, than the other's Antecedent hath to its Conſequent.* Thus the Ratio of 6 to 3 is ſaid to be greater, and the Ratio of 4 to 3 leſs than the Ratio of 5 to 3. Thus again, the Ratio 6 : 3 is greater, and the Ratio 6 : 5 leſs than the Ratio 6 : 4.

112. Hence, when two Ratios are to be compared whoſe Antecedent and Conſequent are both different, *it will be proper to reduce them to the ſame Antecedent, or the ſame Conſequent, before the Compariſon be made*; as for Inſtance, ſuppoſe I would know which of the two Ratios 7 : 5 or 4 : 3 be the greater; to know this, I ſay, as $4 : 3 :: 7 : 5\frac{1}{4}$; then it is evident, 7 : 5 is a greater Ratio than $7 : 5\frac{1}{4}$ (by 111,) and conſequently greater than 4 : 3. Again, ſuppoſe I would compare the Ratios 3 : 4 and 5 : 7; then I ſay, as $3 : 4 :: 5 : \frac{20}{3} = 7 - \frac{1}{3}$; but the Ratio of $5 : 7 - \frac{1}{3}$ is greater than the Ratio of 5 : 7, and therefore the Ratio 3 : 4 is greater than the Ratio of 5 : 7.

113. In any Series of Numbers, 48, 40, 30, 15, *the Ratio of the Extremes is ſaid to be compounded of all the intermediate Ratios*; *viz.* $48 : 15 = 48 : 40 + 40 : 30 + 30 : 15$; which will eaſily appear by placing all Fraction-wiſe, thus; $\frac{48}{15} = \frac{48}{40} \times \frac{40}{30} \times \frac{30}{15} = \frac{48 \times 40 \times 30}{15 \times 40 \times 30}$, for it is plain, ſince 40×30 is in the Numerator and Denominator both, it makes no Alteration in the Value of the Fraction, which therefore is equal to $\frac{48}{15}$.

114. Hence, on the contrary, any Ratio 48 : 15 may be reſolved into any Number of other leſſer Ratios of which it doth conſiſt, as ſo many Parts of the Whole. Thus $\frac{48}{15} = \frac{48}{40} \times \frac{40}{30} \times \frac{30}{15}$; or $\frac{48}{15} = \frac{48}{45} \times \frac{45}{35} \times \frac{35}{15}$. Thus alſo the Ratio of 48 to 5 may be decompounded, or reſolved into any other Number of Ratios, as $\frac{48}{42} \times \frac{42}{36} \times \frac{36}{27} \times \frac{27}{13} \times \frac{13}{5}$, and ſo in any other Caſe.

115. In any two Quantities or Numbers, whoſe Difference is very ſmall in reſpect of the Quantities themſelves, if ſo much be *added to one*, and *ſubtracted from the other*, as ſhall make their *Difference double* or *triple*, or *half*, or a *third Part* of what it was before, then thoſe Quantities or Numbers ſhall be in a *Duplicate* or a *Triplicate*, or a *Subduplicate*, or a *Subtriplicate Ratio* of that they were in before any ſuch Change was made, nearly.

116. Thus let there be two Numbers 10 and 11, whoſe Difference is 1, then if $\frac{1}{2}$ be added to 11, and taken from 10 we have $11\frac{1}{2}$ and $9\frac{1}{2}$, whoſe Difference is 2, double of the former Difference. Now I ſay, the Ratio of $10\frac{1}{2}$ to $9\frac{1}{2}$ is *duplicate* of that of 11 to 10 nearly; for the Ratio of $11\frac{1}{2}$ to $9\frac{1}{2}$ is reſolvable into the Ratios $11\frac{1}{2} : 10\frac{1}{2}$, and $10\frac{1}{2} : 9\frac{1}{2}$ (by 114.) Now the Ratio of $11\frac{1}{2} : 10\frac{1}{2}$ is greater than the Ratio 11 to 10, and the Ratio of $10\frac{1}{2} : 9\frac{1}{2}$ is nearly as much leſs (as will appear from 112.) therefore the Sum of both thoſe Ratios will be nearly equal to twice the Ratio of 11 to 10, that is, $\frac{11\frac{1}{2}}{10\frac{1}{2}} \times \frac{10\frac{1}{2}}{9\frac{1}{2}} = \frac{11}{10} \times \frac{11}{10}$ nearly; for the firſt is $\frac{120,75}{99,75}$, and the latter is $\frac{121}{100}$; which are very nearly equal.

117. Again, if we add 1 to 11 and take it from 10, we ſhall have 12 and 9, whoſe Difference is 3; then will the Ratio 12 : 9 be triplicate, or three Times as great as the Ratio 11 : 10. For $\frac{12}{9} = \frac{12}{11} \times \frac{11}{10} \times \frac{10}{9}$ (by 114) $= \frac{1320}{990}$; and $\frac{11}{10} \times \frac{11}{10} \times \frac{11}{10} = \frac{1331}{1000}$, which two Fractions are very nearly equal.

118. Thirdly, if the Difference between 11 and 10 be reduced to half, or a third Part, the Ratio will be reduced *ſubduplicately* or *ſubtriplicately*; thus add $\frac{1}{4}$ to 10, and take it from 11, and we have $10\frac{1}{4}$ and $10\frac{3}{4}$, whoſe Difference is half of the former. Now $10\frac{1}{4} : 10\frac{3}{4}$ is *ſubduplicate* of the Ratio 10 : 11, or as $\sqrt{10} : \sqrt{11}$. Alſo, if $\frac{1}{3}$ be added to 10 and taken from 11, you have $10\frac{1}{3}$ and $10\frac{2}{3}$, which are nearly in a *ſubtriplicate* Ratio of 10 to 11, or $\sqrt[3]{10}$ to $\sqrt[3]{11}$.

We ſee in theſe Examples how near theſe Ratios come to the Truth, where the Difference is no leſs than a 10th or 11th Part

of

of the Whole; but if we ſuppoſe the Difference to be a much leſs Part of the Whole, as an 100th, a 1000th, &c. they will be much more accurate; ſo that to multiply or divide the Ratio it will be ſufficient to encreaſe or diminiſh one of the Numbers only. Thus 100 : 102 is *duplicate* of the Ratio 100 : 101; and 100 : 103 is *triplicate* of the Ratio of 100 : 101. Alſo, 100 : $100\frac{1}{2}$ is *ſubduplicate*, and 100 : $100\frac{1}{3}$ is *ſubtriplicate* of the Ratio of 100 : 101, nearly.

119. Hence we obſerve, that the Ratio 100 : 101 is to the Ratio 100 : 102 as 1 : 2; and the Ratio 100 : 101 is to the Ratio 100 : 103, as 1 : 3; the Ratio $\overline{100 : 101} : \overline{100 : 104} :: 1 : 4$; and ſo on univerſally, which Theorem is of very great Uſe, and ought to be well remembered by the *mathematical Reader*. What further relates to the Doctrine of *Ratios and Proportion*, will be delivered in Species (in the *algebraic Part*) which will afford a more abſolute and univerſal Speculation of the Nature and Properties thereof than can be obtained from Numbers.

CHAP. X.

Of the Powers of Numbers, and the Extraction of Roots.

120. WHEN any Number is multiplied by itſelf, it is ſaid to be *ſquared*, and the Product is called the *Square* or ſecond Power of that Number; thus $2 \times 2 = 4$, ſo 4 is the Square of 2; and 2 is ſaid to be the *Square Root* of 4; and is thus expreſſed, $\sqrt{4} = 2$. So $7 \times 7 = 49$, and the Square Root of 49 is $\sqrt{49} = 7$, and ſo of any other Numbers.

121. As a Number multiplied by itſelf produces the Square, ſo that Square being multiplied by the ſaid Number, or Root, produces the *Cube*, or third Power. Thus $4 \times 2 = 2 \times 2 \times 2 = 8$, the *Cube* of 2; and $49 \times 7 = 7 \times 7 \times 7 = 343$, the *Cube* of 7. So the *Cube Root* of 8 is $\sqrt[3]{8} = 2$; and the *Cube Root* of 343 is $\sqrt[3]{343} = 7$; and ſo of others.

122. Thus you proceed by a constant Multiplication of every subsequent Product by the same Number to raise any Power of that Number you think fit, as you see done for the 9 Digits in the following Table to the sixth Power.

Root, or first Power.	1	2	3	4	5	6	7	8	9
Square, or second Power.	1	4	9	16	25	36	49	64	81
Cube, or third Power.	1	8	27	64	125	216	343	512	729
Biquadrate, or fourth Power.	1	16	81	256	625	1296	2401	4096	6561
Sursolid, or fifth Power.	1	32	343	1024	3125	7776	16807	32768	59049

123. When the *Square Root* of any Number is proposed to be found or extracted, it is done by the following

RULE.

First, let the Figures of the Number be distinguished into Pairs, by fixing a Point over every other one, beginning at Unit's Place: Then write such a Figure for the first Place in the Root, whose Square shall be equal to, or next less than the Figure or Figures from the last Point on the Left Hand; then subtracting that Square, the other Figures of the Root will be found by taking down each Pair of Figures successively to the Remainders, for new Dividends, and doubling the Root so far as extracted for the first Part of the Divisor, enquire how often it is contained in the new Dividend, and place the Quotient for another Figure in the Root, and also annex it to the Divisor, which will be then compleated.

124. This will be illustrated by the following Examples. Query, the Square Root of 144?

$$
\begin{array}{rl}
 & \dot{1}4\dot{4}\ (\ 12,\ \text{the Square Root required.} \\
 & 1 \\
 & \overline{} \\
22\ ) & 44 \\
 & 44 \\
 & \overline{} \\
 & \cdot\ \cdot
\end{array}
$$

The

The Reaſon of pointing the Number in this Manner is, becauſe there are always *as many Places of Figures in the Root as there are Points over the given Number.* But, till we come to Algebra, the Reaſon of the whole Operation will not ſo well appear as by working it at large (for the common Way is but a Sort of Contraction) in the Manner following.

```
         . .
         144 (10
         100 (  2
               20
   20 )  44 ( 2, then 10 + 2 = 12, the Root.
    2
   22 )  44
         ——
         . .
```

Extract the Root of 219024.

```
                  .  . .
                  219024 ( 400 + 60 + 8 = 468, the Root.
                  160000
                  ——————
          800 )    59024 ( 60
           60
          860 )    51600
                  ——————
  460 × 2 = 920 )   7424 ( 8
              8
            928 )   7424
                  ——————
                    ....
```

But this Example contracted in the common Way ſtands thus.

```
           .  . .
           219024 ( 468, the Root.
           16
           ——
     86 )   590
            516
           ——————
    928 )   7424
            7424
           ——————
            ....
```

Another

Another Example here follows. To extract the Square Root of the Number 29506624.

```
          . . . .
         29506624 ( 5432 = the Root required.
         25
         —
   104) ·450
         416
         ———
  1083) .3466
         3249
         ————
 10862) 21724
        21724
        —————
        .....
```

125. It is plain that Number must be a Square, whose Root may be extracted without a *Remainder*; and such Numbers may be as well *Decimal* as *Integral*; as in the following Example. What is the Square Root of 156,25?

```
        .  .  .
        156,25 ( 12,5, the Root required.
        1
        —
   22) ·56
        44
        —
  245) 1225
       1225
       ————
       ....
```

What is the Square Root of 50,2681?

```
        .  .  .
        50,2681 ( 7,09 = Root required.
        49
        —
  1409 ) 12681
         12681
         —————
         .....
```

Re-

Required the Square Root of 0,366025.

```
 ·  · · ·
0,366025 ( 0,605 = the Root.
0
——
  36
  36
  —
1205 ) · ·6025
         6025
        —————
         ....
```

What is the Square Root of 0,00015625?

```
·  · · · ·
0,00015625 ( 0,0125 = the Root.
       1
       —
 22 ) ·56
       44
       —
245 ) 1225
      1225
      ————
      ....
```

126. The Number that is not a Square, or whose Root cannot be extracted, is said to be *surd*, or *irrational*; but the Roots of such surd Numbers may be approximated in Decimals as near the Truth as required. For Instance, let it be required to extract the Square Root of 2. Annex to the given Surd, as many Pairs of Cyphers as you would have decimal Places in the Root, Thus

```
        . . . . . .
        2,0000000000 ( 1,41421
        1
        —
    24 ) 1 00
           96
           —
   281 ) · ·400
           281
          ————
  2824 ) 11900
         11296
        ———————
 28282 ) · ·60400
            56564
          ————————
282841 ) · 383600
           282841
          ————————
           100759
```

Thus it appears the Square Root of 2 is 1,41421 true to five Places of Decimals; if a greater Degree of Accuracy be required, more Places may be obtained by dividing the Remainder 100759 in the common Way by the Divisor 28284 (omitting the first Figure 1) as you see below.

```
28284 ) 100759 ( ·3562
         84852
        ———————
         159070
         141420
        ———————
         176500
         169704
        ———————
         · ·67960
            56568
          ———————
            11392
```

Therefore the Square Root of 2 is still more truely 1,414213562.

127. The Square Root of any Vulgar Fraction is extracted by extracting the Root of the Numerator and Denominator for the fractional Root. Thus the Square Root of $\frac{25}{36}$ is $\frac{5}{6}$; for 5 is the Root for 25, and 6 the Root of 36, and $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$; and so the Square Root of $\frac{144}{256}$ is $\sqrt{\frac{144}{256}} = \frac{12}{16}$; and so you proceed for any other. But in some Cases, the best Way will be to convert the *Vulgar* into a *Decimal Fraction* (by 91.) and so extract the Root in Decimals, as above taught (in 125,126.)

128. The Extraction of the *Cube Root* will prove too difficult a Task in common Numbers; and as Nothing can be easier by Logarithms, I shall there shew the Method of doing it. Besides, the Reason of the Thing cannot be shewn till the Reader comes to the Algebraic Part, where it will be evident enough.

129. The Biquadrate Root of any Number is easily had by extracting the Square Root of the given Number first, and then the Square Root of that Root: Thus, let it be required to extract the *Biquadrate Root* of 4857532416.

```
           . . . . .
           4857532416, ( 69696,
           36
           --
    129 )  1257
           1161
           ----
   1386 )   9653
            8316
            ----
  13929 )  133724
           125361
           ------
 139386 )  836316
           836316
           ------
           . . . . .
```

Then 69696, (264, = the Biquadrate Root.

```
         4
        ---
  46 )  296
        276
        ----
 524 )  2096
        2096
        ----
        ....
```

For 264 × 264 × 264 × 264 = 4857532416. And thus you proceed in any other Case where the *Biquadrate Root* is required.

CHAP. XI.

Of the Nature and Use of LOGARITHMS.

130. IF a Series of Numbers in *Arithmetical Progression*, beginning from 0, and whose common Ratio is Unity, be appositely placed over another Series of Numbers in *Geometrical Progression*, in the following Manner, *viz.*

Arith.	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	&c.
Geom.	1,	2,	4,	8,	16,	32,	64,	128,	256,	512,	&c.

Then we may observe the following Things.

131. *First*; The several Terms in the Arithmetical Series *expound the Ratio* of the corresponding Terms in the Geometrical Series to the first Term; thus, 2 in the upper Series shews the Ratio of the Geometrical Series is *twice* repeated between 4 and 1; 5 shews the Ratio 5 Times repeated between 32 and 1; or that the Ratio 32 : 1 is 5 Times as great as the Ratio 2 : 1; and 9 denotes the Ratio 512 : 1 to be 9 Times as great as the first Ratio 2 : 1; and so of the Rest.

132. *Secondly*; That to the *Addition* of any two or more Terms in the Arithmetical Series, corresponds a *Multiplication* of the Numbers under them in the Geometrical Series; as in the following Examples.

A.

A. $1+2=3$; $2+3+4=9$; $1+3+5=9$; *&c.*
G. $2\times4=8$; $4\times8\times16=512$; $2\times8\times32=512$; *&c.*

133. *Thirdly*; That for every *Subtraction* of Terms in the Arithmetical Series, there corresponds a *Division* of the Numbers under them in the Geometrical Series; as in these Instances.

A. $2-1=1$; $7-3=4$; $9-4=5$; *&c.*
G. $4\div2=2$; $128\div8=16$; $512\div16=32$; *&c.*

134. *Fourthly*; That when any Term in the Arithmetical Series is *doubled*, *tripled*, *quadrupled*, &c. there answers an *Involution* of the corresponding Term in the Geometrical Series to the *second*, *third*, *fourth*, &c. *Power*, Thus

A. $3+3=6$; $3+3+3=9$; $2+2+2+2=8$.
G. $8\times8=64$; $8\times8\times8=512$; $4\times4\times4\times4=256$.

135. *Fifthly*; That if any Term in the Arithmetical Series be *divided* by 2, 3, 4, *&c.* there answers an Evolution or Extraction of the *Square*, Cubic, Biquadratic, *&c.* Root in the corresponding Term of the Geometrical Series. Thus

A. $4\div2=2$; $6\div3=2$; $8\div4=2$; *&c.*
G. $\sqrt{16}=4$; $\sqrt[3]{64}=4$; $\sqrt[4]{256}=4$; *&c.*

136. *Sixthly*; That in the Arithmetical Series, if any three proximate Numbers be taken, the *Sum of the two Extreams is equal to double the middle Number*; thus, 3, 4, 5, give $3+5=4\times2=8$; and 7, 8, 9 give $7+9=8\times2=16$. Whence also $\frac{3+5}{2}=4$, and $\frac{7+9}{2}=8$, or *half the Sum of the Extremes is equal to the Mean.* And the Case is the same if any two Numbers of the Series be taken, their Sum will be always double of the Mean or middle Term between them. Thus, 1, 3, 5 give $1+5=2\times3=6$; and 2, 5, 8 give $2+8=2\times5=10$. Also if 4 Terms be taken any how, the *Sum of the two Extremes will always be equal to the Sum of the two Means*; thus, 1, 2, 3, 4, give $1+4=2+3=5$; and 3, 5, 7, 9, give $3+9=5+7=12$, *&c.*

137.

137. *Seventhly*; That as every *Addition* of Terms in the *Arithmetical Series* has a correſponding *Multiplication* of Terms in the *Geometrical Series* (by 132,) therefore *the Product of the two Extremes, in any three* Proportionals, *is equal to the Square of the Mean*; thus, in 8, 16, 32 we have $32 \times 8 = 16 \times 16 = 256$; and therefore alſo $\sqrt{32 \times 8} = 16$, *viz.* the *Square Root of the Product of any two Numbers is always a Mean Proportional between them.*

138. *Eighthly*; That of any Terms in the Geometrical Series, *the Product of the two Extremes is ever equal to the Product of the two Means.* Thus, 2, 4, 8, 16, give $16 \times 2 = 8 \times 4 = 32$. And 2, 8, 32, 128, give $128 \times 2 = 32 \times 8 = 256$.

139. *Ninthly*; Hence, of four Terms in the Arithmetical Series, the *laſt is equal to the Difference between the Sum of the two Means and the firſt Term.* Thus, of 3, 5, 7, 9, we have $5 + 7 - 3 = 9$; or $7 + 5 - 9 = 3$. And in the *Geometrical Series*, the *laſt of the four Proportionals is equal to the Product of the two Means divided by the firſt Term.* Thus, of 8 : 32 : 128 : 512, we have $\frac{32 \times 128}{8} = 512$; or $\frac{128 \times 32}{512} = 8$. As we have elſewhere ſhewn (in 105.)

140. From what has been hitherto premiſed, it appears, that the Series of Numbers in Arithmetical Progreſſion, are the LOGARITHMS of the Numbers in the other Series of Geometrical Proportionals; for all that we mean by *Logarithms*, is no more than *ſuch a Sort of Numbers as are artfully contrived to expreſs or expound the Ratios of common natural Numbers, conſidered as Terms in a Scale of Geometrical Proportion.* Now theſe Numbers in Arithmetical Progreſſion, anſwer every Part of this Definition of Logarithms with reſpect to the Series below them (by what was obſerved in 131.) and therefore are their *Logarithms*, i. e. *Exponents of their Ratios*, as the Word imports in its *Greek Etymology.*

141. From hence it appears, that if we conſider *our natural Numbers* as Terms in a Scale of Geometrical Progreſſion, then if ſuch other Numbers were invented and adapted thereto in a Series of Arithmetical Progreſſion, theſe would be *Logarithms* of

of the other; to their *Addition* and *Subtraction* would anſwer a *Multiplication* and *Diviſion* of the reſpective common Numbers (by 132, 133.) Alſo by *doubling* or *tripling* them, we Square or Cube their Numbers; (by 134) or by dividing by 2, 3, &c. you extract the *Square*, *Cubic*, &c. Root of their Numbers, (by 135.)

142. Now ſuch a Table or Canon of *Logarithmic Numbers* has been contrived and compoſed by our late Mathematicians, and are in every one's Hands for Uſe. I ſuppoſe I ſcarce need tell the Reader that Lord *Napier* (a *Scotch* Nobleman) invented, and together with the Aſſiſtance of our Countryman Mr. *Henry Brigs*, calculated and compleated the Canon in preſent Uſe. The Labour of doing this was prodigious in the Way they took for it, but of late, eaſier and more conciſe Methods have been invented, of which more hereafter.*

143. That the Reader may have ſome Idea of the Method they took for this Purpoſe, he muſt conſider that 1, 10, 100, 1000, 10000, 100000, &c. are a Series of Numbers in Geometrical Progreſſion, whoſe common Ratio is 10; and a Series of Numbers in Arithmetical Progreſſion adapted to them as Logarithms will ſtand as below.

A.	0.	1.	2.	3.	4.	5.	&c.
G.	1.	10.	100.	1000.	10000.	100000.	&c.

144. If now we ſuppoſe the common Ratio 10 to be divided into 10000000 equal Parts, or *Ratiunculæ*, then ſince the Logarithm of 1 to 10 is 1, this Logarithm or Unity 1, will alſo be divided into 10000000 equal Parts, or *Decimals*; of which half the Number, *viz.* 0,5000000 will be the Logarithm of the Mean Proportional between 1 and 10, which let us call A.

$$\text{Then will the Logarithms be for } \begin{array}{lcl} \text{Numb.} & & \text{Logarithms.} \\ \left\{\begin{array}{l} 1 \\ \text{A} \\ 10 \end{array}\right. & \begin{array}{c} : \\ : \\ : \end{array} & \begin{array}{l} 0,0000000 \\ 0,5000000 \\ 1,0000000 \end{array} \end{array}$$

145.

* Several other Methods of conſtructing a Canon of Logarithms will be delivered when we come to treat of *Algebra* and *Fluxions*; this we have here given, ſtands firſt in Order, and riſes immediately from the Principles of *common Arithmetic*.

145. In like Manner, a Mean Proportional between A and 10, call B, and its Logarithm will be half the Sum of the Logarithms of A and 10; and so

The Logarithms will be for	A	:	0,5000000
	B	:	0,7500000
	10	:	1,0000000

146. Also between 1 and A, and A and B, you find Mean Proportionals and their Logarithms, and thus you may conceive the Process for every one of the 10000000 Means; among which you'll find *eight Means* which will be so near the same with our eight Digits 2, 3, 4, 5, 6, 7, 8, 9, that the Difference will be wholly inconsiderable, and also their Logarithms, which will be as below, *viz.*

	Numb.		Logarithms.	Numb.		Logarithms.
Thus,	1	:	0,0000000	6	:	0,7781513
	2	:	0,3010300	7	:	0,8450980
	3	:	0,4771213	8	:	0,9030900
	4	:	0,6020600	9	:	0,9542425
	5	:	0,6989700	10	:	1,0000000

147. That is, since there are 10000000 Mean Proportionals between 1 and 10, the Number 2 will be the 3010300th of these; so the Number 3 will be the 4771213th; the Number 5 will be the 6989700th, and so of the Rest. Also the Ratio or Distance of 4 from Unity being twice as great as the Distance of 2, its Logarithm is twice as big, and for the same Reason the Logarithm of 9 is twice as big as the Logarithm of 3; and the Logarithm of 8, three Times as great as the Logarithm of 2; and so on.

148. In like Manner, the Logarithm for all the Numbers between 10 and 100, 100 and 1000, and so on to 100000, denote the Places or Distances of those Numbers in a Scale of Geometrical Proportionals, consisting of 10000000, 20000000, 30000000, &c. Terms. Thus 73 is the 18633229th Term; 743 is the 28709888th Term; and 9745 is the 39887818th Term in the Scale or Series of 40000000.

149. But since we make the Exponent or Logarithm of 1 to 10, to be 1; that of 1 to 100, 2; &c. (by 143) therefore these

these Logarithms must be looked upon as *Decimal Numbers*, and only the first Figure Integral; which is called the Index; and they are thus expressed.

viz.	3	:	0,4771213
	73	:	1,8633229
	743	:	2,8709888
	9745	:	3,9887818

150. And here you observe the Index of the Logarithm is *less by Unity* than the Number of Figures in that Number of which it is the Logarithm, the Reason of which is very plain from (143) and what has been since delivered. N. B. *In reading, this Table of Logarithms should lie before the Eye for Inspection.*

151. If a Number consist of the same Figures, whether it be *integral, mixed*, or *pure Decimals*, the Logarithm will still be the same, except the *Index, which will be always less by* 1, *than the Number of Places in the Integral Part* (149). And when the Number is purely Decimal, the Indexes will also be expressed Decimally, all which will be clear by the Examples in the following Table.

N. B. As many as the Decimal Index is less than 9, so many Cyphers are prefixed to the Decimal Numbers, as is here evident.

Numbers		Logarithms.
9541	:	3,9795939
954,1	:	2,9795939
95,41	:	1,9795939
9,541	:	0,9795939
0,9541	:	,9,9795939
0,09541	:	,8,9795939
0,009541	:	,7,9795939

152. It remains now that we shew how commodiously the Operations of common Arithmetic are performed by Logarithms, and first,

Of Multiplication *by* Logarithms.

The Rule for the Operation is this, *add together the Logarithms of the Factors, the Sum is the Logarithm of the Product.* (By 132) But the Difficulty consists in finding the proper Indices to the Sums; for which observe the following Particulars. (1.) If the Indices are both *integral*, the Sum is so too. (2.) If one be Integral, and the other Decimal, the Sum, if under 10, will

be Decimal; if just 10, or more than 10, cast away 10, and the Remainder is Integral. (3.) If the Indices are both Decimal, and the Sum above 10, cast away 10, and the Remainder will be Decimal; and then the Cyphers to be prefixed to the Decimal Product, will be as many as such an Index is less than 9. See the following Examples.

153. *Examples of* INTEGERS.

Example I.	Multiply	12	=	1,079181
	By	8	=	0,903090
	The Product	96	=	1,982271
Example II.	Multiply	526	=	2,720986
	By	100	=	2,000000
	The Product	52600	=	4,720986
Example III.	Multiply	987600	=	5,994581
	By	517	=	2,713490
	The Product	510589200	=	8,708071

154. *Examples of mixed Numbers.*

Example I.	Multiply	12,4	=	1,0934217
	By	3,6	=	0,5563025
	The Product	44,64	=	1,6497242
Example II.	Multiply	36,5	=	1,5622929
	By	0,00019	=	,6,2787536
	The Product	0,006935	=	,7,8410465
Example III.	Multiply	0,762	=	,9,8819550
	By	570	=	2,7558748
	The Product	434,34	=	2,6378298
Example IV.	Multiply	0,0097	=	,7,9867717
	By	0,00021	=	,6,3222193
	The Product	0,000002037	=	,4,3089910

155.

155. DIVISION of *Integers* and *Decimals* by LOGARITHMS.

The Rule. { From the Logarithm of the *Dividend*, Subtract the Logarithm of the *Divisor*; The *Remainder* is the Logarithm of the *Quotient*.

Example I.	Divide	44,64	=	1,6497242
	By	12,4	=	1,0934217
	Quotient	3,6	=	0,5563025
Example II.	Divide	310	=	2,4913617
	By	4,275	=	0,6309361
	Quotient	72,51457	=	1,8604256
Example III.	Divide	434,34	=	2,6378298
	By	7,62	=	9,8819550
	Quotient	570	=	2,7558748
Example IV.	Divide	,006935	=	,7,8410465
	By	36,5	=	1,5622929
	Quotient	,00019	=	,6,2787536
Example V.	Divide	,000002073	=	,4,3089910
	By	,00021	=	,6,3222193
	Quotient	,0097	=	,7,9867717

156. As the Operation of such Decimal Numbers as contain *single* or *compound Repetends* is most easily performed by *Logarithms*, (see Inst. 90) so we shall here proceed to that Business, having first premised, that if one Number *A*, be to be divided by another *B*, the Quotient will be the same as when Unity is divided by the Number *B*, and the Quotient multiplied by the other *A*; thus 4) 12 ($3 = \frac{1}{4} \times 12$; now if from the Logarithm of Unity or 0, you subtract the Logarithm of any Number, the Remainder is called the *Arithmetical Compliment* of that Number.

Thus from the Logarithm of Unity = 0,0000000
take the Logarithm of 12,4 = 1,0934217

remains the Arithmetical Compliment = ,8,9065783
which is plainly nothing more than the Logarithm of the Fraction $\frac{1}{12,4}$; if therefore

To the Logarithm of	44,64 =	1,6497242
You add the Arithmetical Compliment of	12,4 =	8,9065783
The Sum is the Logarithm of	3,6 =	0,5563025

The same as in Example I. of Division (155).

157. Any Digit multiplied by 10 and divided by 9, becomes a *Repetend*; thus 6 × 10 = 60, and 9) 60 (= 6,666$\dot{6}$, &c. = 6,$\dot{6}$; and the same in Decimals 0,6 × 10 = 6, and 9) 6 (= 0,$\dot{6}$. Also any Number multiplied by Unit with as many Cyphers annexed as it contains Places, and then divided by as many Nines, becomes a *compound Repetend*. Thus 23 × 100 = 2300, and 99) 2300, (= 23,$\dot{2}\dot{3}$, and 527, × 1000, = 527000, and 999,) 527000, (= $\dot{5}2\dot{7}$; and so of others.

158. Hence since the Logarithms of $\frac{10}{9}$, $\frac{100}{99}$, $\frac{1000}{999}$, &c. are 0,0457575, 0,0043648, 0,0004345, &c. therefore we easily obtain the Logarithms of any pure, single, or compound Repetend.

Example. Required the Logarithm of $\dot{6}$?

To the Logarithm in the Table for	6 =	0,7781512
Add the Logarithm of $\frac{10}{9}$	=	0,0457575
The Sum is the Logarithm of	$\dot{6}$ =	0,8239087

And because it may be of Use sometimes, I have here subjoined a Table of the Logarithms of all the Nine Digits *perpetually circulating*.

$\dot{1}$ =	0,0457575
$\dot{2}$ =	0,3467875
$\dot{3}$ =	0,5228787
$\dot{4}$ =	0,6478175
$\dot{5}$ =	0,7447275
$\dot{6}$ =	0,8239087
$\dot{7}$ =	0,8908555
$\dot{8}$ =	0,9488475
$\dot{9}$ =	1,0000000

159. In the same Manner we proceed for the Logarithms of pure compound Repetends.

Example 1. Required the Logarithm of the Repetend $\dot{2}\dot{4}$?

To the Tabular Logarithm of	24, =	1,3802112
Add the Logarithm of	$\frac{100}{99}$ =	0,0043648
The Sum is the Logarithm of	$\dot{2}\dot{4}$, =	1,3845760

Example 2. Required the Logarithm of $\dot{3}6,\dot{5}$?

To the Tabular Logarithm of	$\dot{3}6,\dot{5}$ =	1,5622929
Add the Logarithm of	$\frac{1000}{999}$ =	0,0004345
The Sum is the Logarithm of	$\dot{3}6,\dot{5}$ =	1,5627274.

160.

160. Now with Respect to *mixed Repetends*, as 2,$\dot{6}$, 2,7$\dot{5}\dot{3}$, 7$\dot{2}$5,$\dot{6}$, &c. it is evident (by 157) that 2,$\dot{6}$ = 2,$\frac{6}{9}$, and that 2,7$\dot{5}\dot{3}$, = 2,7$\frac{53}{99}$, and 7$\dot{2}$5,$\dot{6}$ = 7$\frac{25\cdot 6}{999}$ and so on; but 2,$\frac{6}{9}$ = $\frac{24}{9}$, and this multiplied by 9, becomes barely 24; but any Number multiplied by 9, is the same as when multiplied by 10, and once subtracted; whence 2,$\dot{6}$ × 10, — 2,$\dot{6}$ = 24. Hence this *general Rule* is deduced.

From the mixed Repetend multiplied by 10, subtract itself; or (which is the same Thing) from any mixed Repetend subtract the terminate Part, and to the Logarithm of the Remainder, add the Arithmetical Complement of as many Nines as there are Places of Figures in the Repetend, and the Sum will be the Logarithm of the said mixed Repetend.

Example 1. Required the Logarithm of 2,$\dot{6}$?

From the given Repetend	—	2,$\dot{6}$	
Subtract the terminate Part	—	2	
To the Logarithm of the Remainder		2,4	= 0,3802112
Add the Arithmetical Compliment of		9	= 0,0457575
The Sum is the Logarithm of	—	2,$\dot{6}$	= 0,4259687.

Example 2. Required the Logarithm of 2,7$\dot{5}\dot{3}$?

From the given Repetend	—	2,7$\dot{5}\dot{3}$	
Subtract the terminate Part	—	27	
Then to the Logarithm of the Remainder		2,726	= 0,4355258
Add the Arithmetical Complement of the Logarithm of — —		99	= 0,0043648
The Sum is the Logarithm of the Repetend — —		2,7$\dot{5}\dot{3}$	= 0,4398906

Example 3. Required the Logarithm of 7$\dot{2}$5,$\dot{6}$?

From the Repetend —	—	7$\dot{2}$5,$\dot{6}$	
Subduct the constant Part	—	7	
To the Logarithm of the Remainder		724,9	= 2,8602781
Add the Arithmetical Compliment of	—	999	= 0,0004345
The Sum is the Logarithm of	—	7$\dot{2}$5,$\dot{6}$	= 2,8607126.

161. The Learner is now prepared to work any of the common Rules of Arithmetic, where the Operations are tedious or dif-

difficult, in a very easy and concise Manner, by *Logarithms*, of which take the following Examples.

Example 1. In Duodecimal Multiplication.

	F. *In.* *F.* *In.*	
Required the Area of	9 : 10 by 8 : 8 ?	

		F. *In.*	
Add	The Logarithm of	9 : 10 = 9,8$\dot{3}$	= 0,9927008
	To the Logarithm of	8 : 8 = 8,$\dot{6}$	= 0,9378521
		F. *In.*	
Sum is the Logarithm of the Area required.		85 : 2¾ = 85,$\dot{2}$	= 1,9305529.

As *Division* is only the Reverse, it needs no Example.

162. Example 2. In the *Rule of Three Direct*.

If 2 *C.* 3 *Qrs.* 21 *lb.* of Sugar —	2,9375	= 0,4679778
Cost 6*l.* 1*s.* 8*d.* —— ——	6,08$\dot{3}$	= 0,7841316
What will 12 *C.* 2 *Qrs.* cost ? ——	12,5	= 1,0969100
		1,8810416
Answer, 25*l.* 17*s.* 8*d*¾. ——	25,8864	= 1,4130638.

Note, here the Logarithm of the first Number is subtracted from the Sum of both the others ; but a more concise Way is to take the *Arithmetical Complement* of the first (which we denote by *A. C.*) and then the Whole is performed by *one Addition*.

Example 3. At one Operation.

If ½ *C.* of Tobacco (*A. C.*) ——	0,5	= 0,3010300
Cost 4*l.* 17*s.* 8*d.* ——	4,6$\dot{3}$	= 0,6658935
What will 7 *lb.* cost ? ——	0,0625	= ,8,7958800
Answer, 11*s.* 7*d.* ——	0,5791$\dot{6}$	= ,9,7628035.

163. Example 4. *Inverse Proportion*, at one Operation.

If Wheat be 6*s.* 4*d.* *per* Bushel ——	6,$\dot{3}$	= 0,8016325
And the Penny White Loaf weigh 7¾ *oz.*	7,75	= 0,8893017
What is the Weight, at 3*s.* 10*d.* *p.* Bushel (*A. C.*) —— ——	3,8$\dot{3}$	= 9,4164234
Answer, 12 *Oz.* 16 *Pwt.* 2 *Grs.* —	12,8043	= 1,1073576.

Here

Here we take the Arithmetical Complement (*A. C.*) of the third Term, the Reason of which is evident from (156).

164. Extraction of Roots by *Logarithms*.

Example 1. Required the *Square Root* of 2830,24?

The Logarithm thereof is	——	3,4518232
One Half is the Logarithm of the Root	53,2 =	1,7259116

Example 2. What is the *Square Root* of 14,6?

The Logarithm of that Number is	——	1,1663314
Half of which is the Logarithm of the Root	3,8297 =	0,5831657

165. Example 3. Required the Cube Root of 1,728?

The Logarithm of — —	1,728 =	0,2375437
A third of which is the Logarithm of the Root	1,2 =	0,0791812.

Example 4. What is the Cube Root of 0,27589?

The Logarithm thereof is	——	,9,4407132
$\frac{1}{3}$ Part thereof is the Logarithm of	0,6509 =	,9,8135710.

N. B. In Decimal Numbers, where the Indices are Decimal, in extracting the *Square Root* you add 10 to the Index; for the *Cube Root* 20; for the *Biquadrate Root* 30; and so on.

Example 5. Required the several Roots of the last Number, for Instance.

The Logarithm of the Number	0,27589	=,9.4407132
The Half is the Logarithm of the *Square* Root — —	0,52523	=,9.7023566
A third Part, of the *Cube* Root —	0,6509 &	=,9.8135710
A Fourth, the *Biquadrate* ——	0,7247 &	=,9.8601783
A Fifth, the *Surfolid* Root — —	0,7729 &	=,9.8881426

Thus you see how extremely easy it is to extract any Root out of any Number by Logarithms, and especially in such Cases where by the common Rules the Operation is very laborious and difficult, and sometimes quite impracticable.

The Reason of extracting Roots by Logarithms, is evident from what we have observed of these Numbers in (135). Since to Square or Cube any Number, is only to multiply it by it's self once,

once, or twice, we thought it needless to give any Examples in Logarithms of that Affair; especially, since they are only the Reverse of the foregoing, and evident from (134.)

166. To find a *Mean Proportional* between any two Numbers, as 3, and 243.

Add the Logarithm of the Numbers	3, =	0.4771212
	243, =	2.3856063
The Sum is — — —	=	2,8627275
Half of which is the Logarithm of the Mean required, *viz.*	27, =	1,4313637

For 3 : 27 :: 27 : 243.

167. To find two Mean Proportionals between any two Numbers, as 27, and 729.

To double the Logarithm of the first Number — — — —	27, =	1,4313638
		1,4313638
Add the Logarithm of the last —	729, =	2,8627275
The Sum is — — —		5,7254551
A Third of which is the Logarithm of the *first Mean*, viz.	81, =	1,9084850
To this add the Logarithm of the last Number again, *viz.* —	729, =	2,8627275
The Sum is — — —		4,7712125
Half that is the Logarithm of the second Mean required. —	243, =	2,3856062

For 27 : 81 :: 243 : 729. Or thus, having found the *first Mean*, as here 81, you know then the *Ratio* of the Series 27) 81 (= 3 the Ratio, therefore *from the Logarithm of the first Mean, subtract the Logarithm of the first given Number, the Remainder is the Logarithm of the Ratio; which add to the Logarithm of the first Mean, the Sum is the Logarithm of the second Mean; and again, added to the Logarithm of the Second, it gives the Logarithm of the third Mean*, and so on. All which is evident from what we first premised concerning the Nature of a Geometrical Series, and Logarithms adapted thereto. See (130, 131, *&c.*)

We might now have proceeded to the Computations of *Interest*, *Annuities*, &c. which are best performed by Logarithms, but

but as the *Theorems* or *Canons* for this Purpoſe muſt be raiſed by an *algebraic* Proceſs, we muſt defer this Buſineſs till we treat of that Science in the next enſuing Part. Alſo the Uſe of Logarithms in *Navigation*, and other mathematical Arts, will be largely ſhewn when we come to treat of thoſe Subjects. What we have done at preſent being ſufficient for all the common Parts of Arithmetic, where *numerical Calculations* only are concern'd.

And as to the *natural* or *hyperbolical Logarithms* we ſhall refer them to the Doctrine of Fluxions, where we ſhall ſhew how they are made, and how theſe we here treat of are derived from them: Alſo *logiſtical Logarithms* will be fully treated of in the practical Part of Aſtronomy, where they are uſed; and *Tables* of every Sort of Logarithms will be ſupplied in their proper Places.

INSTITUTIONS OF ALGEBRA.

CHAP. I.

168. ALGEBRA is a Kind of *ſpecious Arithmetic*, as we here make uſe of *Species*, i. e. Symbols or Letters to repreſent Quantities of every Kind, as well known as unknown. And it is cuſtomary to repreſent *known Quantities* by the firſt Letters of the Alphabet, *a*, *b*, *c*, *d*, &c. and *unknown* ones by the laſt Letters, *x*, *y*, *z*.

169. The Terms of an algebraic Expreſſion are connected under all their various Relations by proper Symbols or Characters, as mentioned in (22.) but we are here to conſider the two principal Symbols, + and —, in a more extenſive View than we did there, for they are deſigned to repreſent any two contrary Modes, Qualities, or Actions, &c. In ſhort, what ever is repreſented by the *affirmative Sign* +, as $+a$; the Contrary is repreſented by the *negative Sign* —, as $-a$.

Thus if $+a$ signifies any Thing	added above, to the Right, forwards,	$-a$ signifies	subtracted below, to the Left, backwards.

If $+a$ signifies any	Increase, Gravity, Money due, Motion upwards,	$-a$ signifies any	Decrease, Levity, Money owing, Motion downward

And so on in every Kind of Contrariety. And two such Quantities connected together in any Case destroy each other's Effect, or are equal to nothing, as $+a-a=o$. Thus, if a Man has but 10*l.* and at the same Time owes 10*l.* he is worth *nothing*.

170. If the first Quantity or Term be affirmative we neglect the Sign $+$, as $a+b$, instead of $+a+b$. If an algebraical Quantity consist of two Terms, it is called a *Binomial*, as $a+b$; if of three Terms, a *Trinomial*, as $a+b+c$. If there be more Terms, it is called a *Multinomial*. All which are *compound* Quantities. *Simple* Quantities consisting of one Term only, as $+a$, $+ab$, $+abc$.

171. When simple Quantities are to be multipled together we do not, generally, use the Symbol $\times$, but place them together without; as ab instead of $a\times b$, and abc for $a\times b\times c$; when compound Quantities are to be represented as multipled, then a Line is drawn over the Factors connected with the proper Sign $\times$; as $\overline{a+b}\times c$, $\overline{a+x}\times\overline{b-y}$.

172. When any Number of Figures are prefixed to the Terms of an algebraic Quantity, they are called *Coefficients*, and shew how often the Quantity is to be taken; as $3a$, $5bc$, $2a+5b$, *&c.* and when no Number is prefixed, *Unity* is always understood, tho' not expressed, for the *Coefficient*; thus a is the same as $1a$, and bc as $1bc$, since any Thing multiplied by *Unity* is still the same.

173. Quantities are said to be *like*, or *similar*, that are represented by the same Letter or Letters equally repeated; thus $+3a$ and $-5a$ are *like*; but a and b, or a and aa are *unlike*. What other Definitions and Symbols are used in *Algebra* will be explained in their proper Places.

CHAP. II.

ADDITION.

174. IN the ADDITION of algebraic Quantities there are three CASES, as follow.

CASE I. To add Quantities that are *like*, and have *like Signs*.

RULE.

Add together the Coefficients, to their Sum subjoin the common Quantity, or Letters, and prefix the Sign where necessary.

EXAMPLES.

$$\begin{array}{lr}
\text{To} & 5a \\
\text{Add} & 3a \\ \hline
\text{Sum} & 8a
\end{array}
\qquad
\begin{array}{r}
3a - 5b \\
a - 2b \\ \hline
4a - 7b
\end{array}
\qquad
\begin{array}{r}
a + 2b - 3c \\
3a + b - 4c \\ \hline
4a + 3b - 7c
\end{array}$$

$$\begin{array}{lr}
\text{To} & ab - 5b + 3x - 21y + 15z + 7 \\
\text{Add} & 4ab - b + 7x - 4y + 9z + 10 \\ \hline
\text{Sum} & 5ab - 6b + 10x - 25y + 24z + 17
\end{array}$$

175. CASE II. To add Quantities that are *like*, but have *unlike Signs*.

RULE.

Subtract the lesser Coefficient from the greater, to the Remainder prefix the Sign of the greater, and subjoin the common Letter or Quantities.

EXAMPLES.

$$\begin{array}{lr}
\text{To} & -4a \\
\text{Add} & +7a \\ \hline
\text{Sum} & +3a
\end{array}
\qquad
\begin{array}{r}
-5b - 6c - 9x \\
+3b + 8c + 5x \\ \hline
-2b + 2c - 4x
\end{array}$$

$$\begin{array}{lr}
\text{To} & a + 6x - 5y + 8 \\
\text{Add} & -5a - 4x + 4y - 3 \\ \hline
\text{Sum} & -4a + 2x - y + 5
\end{array}
\qquad
\begin{array}{r}
2a - 2b + 13 \\
-2a + 2b - 10 \\ \hline
0 \quad 0 + 3
\end{array}$$

We proceed here according to Custom, but it is with some Impropriety that we talk of *adding* Quantities with unlike Signs,

 since

ſince the Operation does wholly conſiſt in *Subtraction*, as it muſt from the Nature of the Signs (169, 173.)

176. CASE III. To add Quantities that are unlike.

RULE.

Set them all down one after another, with their Signs and Coefficients prefixed.

EXAMPLES.

To	$2a$	$3a$	$2a - 5bc$
Add	$3b$	$-4x$	$2x + 7$
Sum	$2a + 3b$	$3a - 4x$	$2a - 5bc + 2x + 7$

To	$4a + 4b + 3c - 9$
Add	$-4x - 4y + 3z$
Sum	$4a + 4b + 3c - 4x - 4y + 3z - 9$

CHAP. III.

SUBTRACTION.

177. SINCE the Sign — is juſt oppoſite to the Sign +, and *Subtraction* juſt contrary to *Addition*, therefore to ſubtract a Quantity is the very ſame Thing as to *add* the ſame Quantity with the contrary Sign. Or thus,

If I am to receive 3*l.* - - - - - - -	$+3a$
or to pay 3*l.* - - - - - - - -	$-3a$
The Odds or Difference to me is - - - -	$+6a$
Or, if this Day I have 10*l.* - - - - -	$+10a$
And Yeſterday I owed 3*l.* more than I had to pay -	$-3a$
The Odds in my Fortune is 13*l.* better To-day than Yeſterday - - - - -	$+13a$

Whereas it would have been but 10*l.* better, had not the negative 3*l.* been ſubducted. Therefore we have a

GE-

General Rule.

178. *Change the Signs of the Quantity to be subtracted*, and then add them both together, by the Rules of the preceding Chapter; the Sum arising by such Addition is the true Remainder.

Examples.

From	$+8a$	$4a-7b$
Take	$+3a$	$3a-5b$
Remains	$8a-3a=5a$	$a-2b$

From	$3a$	$-2b+2c-4x$
Take	$-4a$	$-5b-6c-9x$
Remains	$+7a$	$3b+8c+5x$

From	3	$2a-3x+5y-6$
Take	$2a-2b+13$	$6a+4x+5y+4$
Remains	$-2a+2b-10$	$-4a-7x\quad 0-10$

CHAP. IV.

MULTIPLICATION.

179. IN Multiplication there is one general Rule for the Signs, viz. *When the Signs of the Factors are like, (that is,* both $+$, or both $-$) *the Sign of the Product is* $+$; *but when the Signs of the Factors are unlike, the Sign of the Product is* $-$. This general Rule will resolve itself into *four particular Cases*, which we shall illustrate separately in simple Quantities.

180. Case I. When any positive Quantity, as $+a$, is multiplied by a positive Number $+n$, the Meaning is, that $+a$ is to be taken so many Times as there are Units in n; and the Product is evidently n Times a, or na.

Examples.

Multiply	$+a$	$2a$	$5bx$	$9dc$
By	$+n$	$3b$	7	$4y$
Product	na	$6ab$	$35bx$	$36ydc$

Case

181. Case II. When $-a$ is multiplied by n, then $-a$ is to be taken as often as there are Units in n, and the Product must be n Times $-a$, or $-na$.

Examples.

Multiply	$-a$	$-2a$	$-5bx$	$-9dc$
By	$+n$	$3b$	7	$4y$
Product	$-na$	$-6ab$	$-35bx$	$-30dcy$.

182. Case III. As Multiplication by a *positive Number* implies a *repeated Addition*, so Multiplication by a *Negative* implies a *repeated Subtraction*; and therefore when $+a$ is to be multiplied by $-n$, it means only that $+a$ is to be subtracted as often as there are Units in n; and therefore the Product being *negative* must also be $-na$.

Examples.

Multiply	$+a$	$2a$	$5bx$	$9dc$
By	$-n$	$3b$	-7	$-4y$
Product	$-na$	$-6ab$	$-35bx$	$-36dcy$.

183. Case IV. When $-a$ is to be multiplied by $-n$, then $-a$ is to be subtracted as often as there are Units in n; but to subtract $-a$ is equivalent to *adding* $+a$, (177.) therefore this Case is the same in Effect with the Product, and is evidently $+na$.

Examples.

Multiply	$-a$	$-2a$	$-5bx$	$-9dc$
By	$-n$	$-3b$	-7	$-4y$
Product	$+na$	$+6ab$	$+35bx$	$+36dcy$.

184. When the Factors are one or both *compound Quantities*, or consist of several Parts; you must *multiply every Part of the Multiplicand by each Part of the Multiplier; and then add all the Products into one Sum; and that Sum shall be the Product required.*

Exam-

EXAMPLES.

$$
\begin{array}{llll}
\text{Mult.} & a+b & a-b+c & a-2y+5z-6 \\
\text{By} & a & -b & 8 \\
\hline
\text{Prod.} & aa+ab & -ab+bb-bc & 8a-16y+40z-48 \\
\hline
\end{array}
$$

$$
\begin{array}{lll}
\text{Multiply} & 4a+b & 2a-4b \\
\text{By} & a-2b & 2a+4b \\
\hline
 & 4aa+ab & 4aa-8ab \\
 & \quad -8ab-2bb & \quad 8ab-16bb \\
\hline
\text{Product} & 4aa-7ab-2bb & 4aa \quad 0 \quad -16bb \\
\hline
\end{array}
$$

$$
\begin{array}{lll}
\text{Mult.} & xx-ax & aa+ab+bb \\
\text{By} & x+a & a-b \\
\hline
 & xxx-axx & aaa+aab+abb \\
 & \quad +axx-aax & \quad -aab-abb-bbb \\
\hline
\text{Prod.} & xxx \quad 0 \quad -aax & aaa \quad 0 \quad 0 \quad -bbb \\
\hline
\end{array}
$$

185. I shall here insert one Example more, to shew the Reason of the common Method of proving the Work of Multiplication in Numbers, *by casting out the Nines in the Factors, and Product of their Remainders*, and also out of the *general Product, to observe the Equality of the Remainders.* (See *Inst.* 55.) We shall take the first Example in (*Inst.* 54,) where $1750 \times 76 = 133000$. Therefore

$$
\begin{array}{lll}
\text{Mult.} & 9a+c & =1750=\overline{9\times194}+4. \\
\text{By} & 9b+d & =\quad 76=9\times 8\ +4. \\
\hline
 & 81ab+9bc & 10500 \\
 & \quad +9ad+cd & 12250 \\
\hline
\text{Prod.} & 81ab+9bc+9ad+cd & =133000
\end{array}
\quad
\left\{
\begin{array}{l}
a=194 \\
c=\ \ 4 \\
b=\ \ 8 \\
d=\ \ 4
\end{array}
\right.
$$

Now

Now here we are to obſerve, that any Number divided by 9 leaves the ſame Remainder as when the Figures of that Number are added together, and the Nines caſt out as often as they occur; thus $76 \div 9$, leaves 4; and $7 + 6 = 9 + 4$. Alſo $1750 \div 9$ leaves 4, and $1 + 7 + 5 + 0 = 9 + 4$: Laſtly, $133000 \div 9$ leaves 7, and $1 + 3 + 3 = 7$. And thus it will be for every other Number.

Again, 'tis evident, that the Sum of the three firſt Terms of the *algebraic Product* divided by 9 leaves no Remainder, what Remainder therefore is, muſt be from the fourth Term $c\,d$ divided by 9; but this Term is always the Product of the two Remainders of the Factors, c and d; conſequently, if *the Product of theſe Remainders, divided by Nine, leave the ſame Remainder as the Figures of the Product of the two Factors when added together, and the Nines caſt out, the Work will be right*; provided no Error be committed that amounts to Nine, or any Multiple of Nine. *N. B.* I have inſerted this Demonſtration of the Proceſs here, as it is an *algebraic one*, and what has been deſired by many Perſons, who have ſought for it in vain in Books of *Arithmetic* hitherto publiſhed.

CHAP. V.

DIVISION.

168. IN Division of *algebraic Quantities* the Rule for the Signs is the ſame as in *Multiplication*, viz. *If the Signs of the Diviſor and Dividend are like, the Sign of the Quotient muſt be* +; *but if they are unlike, the Sign of the Quotient muſt be* −. This is evidently deduced from the Rule in *Multiplication* (179.) if it be conſidered, *that the Quotient muſt be ſuch a Quantity as multiplied by the Diviſor, ſhall give the Dividend.*

And this is a General Rule for all Operations in *Diviſion*, which are only the Reverſe of *Multiplication*, and will be eaſy to underſtand when illuſtrated by Examples, as follow.

a)

$$a)\,na\,(n; \qquad -a)-na\,(+n; \qquad -a)+na\,(-n.$$

$$
\begin{array}{l}
2a)\,6ab\,(3b; \\
6ab \\
\overline{} \\
\cdot\,\cdot
\end{array}
\qquad
\begin{array}{l}
a)\,aa+ab\,(a+b \\
aa \\
\overline{} \\
\cdot\,\cdot\,ab \\
ab \\
\overline{} \\
\cdot\,\cdot
\end{array}
$$

$$
\begin{array}{l}
4a+b)\,4aa-7ab-2bb\,(a-2b \\
4aa+ab \\
\overline{} \\
-8ab-2bb \\
-8ab-2bb \\
\overline{} \\
\cdot\,\cdot\cdot\,\cdot
\end{array}
$$

$$
\begin{array}{l}
2a-4b)\,4aa-16bb\,(2a+4b \\
4aa-8ab \\
\overline{} \\
+8ab-16bb \\
+8ab-16bb \\
\overline{} \\
\cdot\,\cdot\cdot\,\cdot
\end{array}
$$

$$
\begin{array}{l}
a-b)\,aaa-bbb\,(aa+ab+bb \\
aaa-aab \\
\overline{} \\
+aab-bbb \\
+aab-abb \\
\overline{} \\
+abb-bbb \\
+abb-bbb \\
\overline{} \\
\cdot\,\cdot\cdot\,\cdot
\end{array}
$$

$$
\begin{array}{l}
3a - 6)\ 6aaaa - 96\ (\ 2aaa + 4aa + 8a + 16 \\
\quad\quad\quad 6aaaa - 12aaa \\
\hline
\quad\quad\quad\quad\quad\quad 12aaa - 96 \\
\quad\quad\quad\quad\quad\quad 12aaa - 24aa \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad 24aa - 96 \\
\quad\quad\quad\quad\quad\quad\quad\quad 24aa - 48a \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 48a - 96 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 48a - 96 \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cdot\cdot \quad\quad \cdot\cdot
\end{array}
$$

187. In dividing, when you come to a Remainder of one Term, it is commonly ſet down with the Diviſor under it, after the other Terms, which together make the whole Quotient. Thus

$$
\begin{array}{l}
a + x)\ aa + xx\ (\ a - x + \dfrac{2xx}{a + x} \\
\quad\quad aa + ax \\
\hline
\quad\quad\quad - ax + xx \\
\quad\quad\quad - ax - xx \\
\hline
\quad\quad\quad\quad\quad 2xx
\end{array}
$$

188. It ſometimes happens, that the ſame Letter or Quantity is found in all the Terms of the Diviſor and Dividend, and that there is obviouſly ſome common Meaſure to the Coefficients of the Terms; when this is the Caſe, you expunge the common Quantity, divide by the common Meaſure, and place the Diviſor ſo reduced, under the new Dividend in the Quotient. Thus

$2b)\ ab + bb\ \left(\dfrac{a+b}{2}\right.$, here Unity, or 1 is the common Meaſure. Again,

$$20ad)\ 10ab + 15ac\ \left(= \frac{2b + 3c}{4d}\right.$$

Thus $12ab)\ 30ax - 54ay\ \left(= \dfrac{5x - 9y}{2b}\right.$

And $4aa)\ 8ab + 6ac\ \left(= \dfrac{4b + 3c}{2a}\right.$

CHAP.

CHAP. VI.

Of *FRACTIONS*.

189. ALgebraic *Fractions* are of the ſame Nature, and require the ſame Management as thoſe of Numbers, for ſuppoſe $a = 2$, and $b = 3$; then $\frac{2}{3} = \frac{a}{b}$, a *proper Fraction*; or $\frac{3}{2} = \frac{b}{a}$, an *improper Fraction:* and $2\frac{2}{3} = a + \frac{a}{b}$ a mixed fractional Quantity.

190. A *mixed Quantity* is reduced to an improper Fraction by the Rule in (Inſt. 68.) *viz. Multiply the integral Part by the Denominator of the fractional Part, to which Product add the Numerator; and the Sum will be a new Numerator, under which write the Denominator, and it is the improper Fraction required.* Thus $a + \frac{a}{b}$ becomes $\frac{ab+a}{b}$; $a + b + \frac{x}{z}$ is $\frac{az + bz + x}{z}$; and $a - x + \frac{aa - ax}{x} = \frac{ax - xx + aa - ax}{x} = \frac{aa - xx}{x}$; ſo $a - x + \frac{2xx}{a+x} = \frac{aa + xx}{a + x}$.

191. Fractions of different Denominations are reduced to the ſame Denomination thus; *Multiply all the Denominators together for a common Denominator, and each Numerator by every Denominator but its own, for a new Numerator.* (See Inſt. 69.) So $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}$ will become $\frac{acd}{bcd}, \frac{bbd}{bcd}, \frac{ccb}{bcd}$. Thus $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{edb}{bdf} = \frac{adf + cbf + edb}{bdf}$.

192. A Fraction is reduced to its loweſt Terms by the Rule (in Inſt. 66.) for finding a *common Diviſor*; thus the Fraction $\frac{15ab}{81bx}$ has its common Meaſure, 3 for the Coefficient, and b for the other Part; and ſo the Whole is $3b$, by which dividing the Fraction, it is reduced to its loweſt Terms of the ſame Value, *viz.*

$\frac{5a}{27x} = \frac{15ab}{81bx}$. Thus $\frac{bbc + bbd}{cd + dd} = \frac{b^2}{d}$; and $\frac{aaa + bbb}{aa - bb} = \frac{aa - ab + bb}{a - b}$, being divided by $a + b$ the common Measure of the Numerator and Denominator. *N. B.* It often happens that the same Letter or Letters are contained in every Part or Term of the Fraction, which in such a Case are to be expunged, and the Fraction is thereby reduced to a more simple Form of the same Value. Thus $\frac{ab + db}{bx} = \frac{a + d}{x}$; and $\frac{25za}{5xz + 15az} = \frac{5a}{x + 3a}$ by expunging z, and dividing by 5.

193. When Fractions are to be *added*, *subtracted*, *multiplied*, or *divided*, they should be first reduced to one Denomination, and in their lowest Terms, and then the Rules for the Operations are the same as for *numeral Fractions*. Thus to add $\frac{a}{b}$ to $\frac{c}{d}$, you reduce them to a common Denomination $\frac{ad}{bd}$, and $\frac{bc}{bd}$, then their Sum is $\frac{ad + bc}{bd}$ and $\frac{a}{b} + \frac{c}{d} + \frac{d}{e} = \frac{ade + bce + ddb}{bde}$.

194. To *subtract* one Fraction from another, you reduce them to a common Denominator, and then take the Difference of the Numerators, and under-write the common Denominator. Thus if from $\frac{a}{b}$ you take $\frac{c}{d}$, the Difference is $\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$; from the Integer a take the Fraction $\frac{b}{c}$, the Difference is $a - \frac{b}{c}$, from $\frac{a + x}{b}$ take $\frac{a - x}{c}$, there remains $\frac{ac + cx}{bc} - \frac{ab - bx}{bc}$, that is, $\frac{ac + cx - ab + bx}{bc}$.

195. To *multiply* one Fraction by another, *you multiply the Numerators one into the other for the Numerator of the Product, and the Denominators multiplied, one into another, give the Denominator of the Product.* Thus $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$; and $\frac{a + b}{c} \times \frac{a - b}{d} = \frac{aa - bb}{cd}$.

$\frac{aa-bb}{cd}$. Also $\overline{a+\frac{b}{c}}\times\frac{d}{e}=\frac{\overline{ac+b}}{c}\times\frac{d}{e}=\frac{acd+bd}{ce}$.

$a\times\frac{b}{c}=\frac{a}{1}\times\frac{b}{c}=\frac{ab}{c}$; for any Integer, a, is reduced to the Form of a Fraction by writing Unity under it, as $\frac{a}{1}$.

196. To divide one Fraction by another, *multiply the Numerator of the Dividend by the Denominator of the Divisor, the Product will be the Numerator of the Quotient. Then the Denominator of the Dividend, multiplied by the Numerator of the Divisor, gives the Denominator of the Quotient.* Thus $\left.\frac{a}{b}\right)\frac{c}{d}\left(=\frac{cb}{ad}\right.$; and $\left.\frac{d}{e}\right)\frac{acd+bd}{ce}\left(=\frac{acde+bde}{cde}=\frac{ac+b}{c}=a+\frac{b}{c}\right.$. And $\left.\frac{a-b}{d}\right)\frac{aa-bb}{cd}\left(=\frac{a+b}{c}\right.$, after Reduction; and $\left.\frac{a+b}{a-b}\right)\frac{a-b}{a}\left(=\frac{aa-2ab+bb}{aa+ab}\right.$.

197. In order to demonstrate the Truth, or shew the Reason, of the foregoing Rules for the *Addition*, *Subtraction*, *Multiplication* and *Division* of Fractions, we must here premise the following *Axioms*, or *Principles* that are *in themselves evident Truths*; these will be also necessary in most of our future Mathematical Speculations.

198. AXIOM I. *Things that are equal to one and the same Thing, are equal to one another.*

Thus, if $a=m$, and $b=m$, then $a=b$.

199. AXIOM II. *If to equal Things, you add equal Things, the Sums will be equal.*

Thus, if to the equal Quantities . . . . $a=m$
You add the equal Quantities . . . . . . $b=n$

The Sums will be equal . . . . $a+b=m+n$.

200. AXIOM III. *If from equal Things* . . $a=m$
You subduct equal Things . . . . . . $b=n$

The Remainders will be equal, viz. $a-b=m-n$.

201.

201. AXIOM IV. *If equal Things* . . . $a = m$
Be multiplied by equal Things . . $b = n$

The Product will be equal . . . . $ab = mn$.

202. AXIOM V. *If equal Things* . . . . $a = m$
Be didided by equal Things . . . . $b = n$

The Quotients will be equal, viz. $\frac{a}{b} = \frac{m}{n}$.

203. Now from hence we prove that Fractions of any Kind reduced to the ſame Denomination are *added*, by *adding their Numerators, and ſubſcribing the common Denominator*. Thus $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ for let $\frac{a}{b} = m$, $\frac{c}{b} = n$, and multiplying both Sides of each Equation by b, we have $a = bm$, and $c = bn$ (by Inſt. 201) and $mb + bn = a + c$, (Inſt. 199); and $m + n = \frac{a+c}{b}$ (Inſt. 202.); that is, (by ſubſtituting the Value of m and n) $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. And in the ſame Manner it is ſhewn, that $\frac{a}{b} - \frac{c}{b} = m - n = \frac{a-c}{b}$.

204. Again, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ is thus demonſtrated. Let $\frac{a}{b} = m$, $\frac{c}{d} = n$; then $a = mb$, and $c = nd$; and $bdmn = ac$ (Inſt. 201.) therefore $\frac{ac}{bd} = m \times n = \frac{a}{b} \times \frac{c}{d}$.

205. Laſtly, it is ſhewn that $\frac{a}{b}$ divided by $\frac{c}{d}$ give $\frac{ad}{cb}$; for $mb = a$, and $mbd = ad$ (Inſt. 201); alſo $nd = c$, and $ndb = cb$; therefore $\frac{mbd}{ndb} = \frac{ad}{cb} = \frac{m}{n} = \frac{a}{b} \div \frac{c}{d}$.

CHAP.

CHAP. VII.

Of INFINITE SERIES.

206. WHEN it happens in Division, that the Divisor is not exactly contained in the Dividend, the Operation may be continued without End; and the Quotient will in that Case be an *Infinite Series* of Terms. This will be the Case, if you divide *Unity* by $1 - a$; as below.

$$
\begin{array}{l}
1 - a)\ 1 \quad (1 + a + aa + aaa, \text{ \&c.} \\
\quad\ 1 - a \\
\quad \overline{\quad + a} \\
\quad\quad + a - aa \\
\quad\quad \overline{\quad\quad + aa} \\
\quad\quad\quad + aa - aaa \\
\quad\quad\quad \overline{\quad\quad + aaa} \\
\quad\quad\quad\quad + aaa - aaaa \\
\quad\quad\quad\quad \overline{\quad\quad\quad + aaaa}, \text{ \&c.}
\end{array}
$$

N. B. Here it is soon to be observed, in what Order or Manner the several Terms of the Series in the Quotient will arise, without farther Operation; this is called *discovering the Law of the Series.*

207. Let it be required to divide $aa + xx$ by $a + x$; thus

$$
\begin{array}{l}
a + x)\ aa + xx\ (a - x + \dfrac{2xx}{a} - \dfrac{2x^3}{a^2} + \dfrac{2x^4}{a^3}, \text{ \&c.} \\
\quad\quad aa + ax \\
\quad\quad \overline{- ax + xx} \\
\quad\quad - ax - xx \\
\quad\quad \overline{\quad\quad + 2xx} \\
\quad\quad\quad + 2xx + \dfrac{2xxx}{a} \\
\quad\quad\quad \overline{\quad\quad - \dfrac{2xxx}{a}} \\
\quad\quad\quad\quad - \dfrac{2xxx}{a} - \dfrac{2xxxx}{a^2} \\
\quad\quad\quad\quad \overline{\quad\quad + \dfrac{2xxxx}{a^2}}, \text{ \&c.}
\end{array}
$$

Here

Here the Law of the Series is discovered in a few Terms; and the Series may be continued at Pleasure without farther Trouble; also the Signs are here alternately + and —.

208. Another Example take as follows. Divide ay by $1 + x$.

$$1 + x\,)\,ay\quad(ay - ayx + ayxx - ayxxx, \text{ \&c.}$$

$$\begin{array}{rrrr}
ay + ayx & & & \\
\hline
- ayx & & & \\
- ayx - ayxx & & & \\
\hline
& + ayxx & & \\
& + ayxx + ayxxx & & \\
\hline
& & - ayxxx - ayx^4 & \\
& & - ayxxx - ayx^4 & \\
\hline
& & & + ayx^4, \text{ \&c.}
\end{array}$$

Therefore $\frac{ay}{1+x} = ay - ayx + ayxx - ayxxx$, &c. $= ay \times \overline{1 - x + xx - xxx}$, &c. but also $\frac{ay}{1+x} = ay \times \frac{1}{1+x}$; hence $\frac{1}{1+x} = 1 - x + xx - xxx$, &c. as is evident by dividing each Side of the Equation by ay.

209. This Method of expressing a Fraction in an *Infinite Series* will be often found very advantageous in approximating the Values of Mathematical Quantities expressed in an Algebraic or Fluxionary Manner, as we shall find in many Instances as we proceed. And indeed this is the Foundation of the *Arithmetic of Infinities*; for these interminate Series may be *added* and *subtracted*; *multiplied* and *divided*; *squared* and *cubed*; and the *Square* and *Cube Roots* extracted; and so that the *Sum* and *Difference*; *Product* and *Quotient*; *Power* and *Root*, shall still be an *infinite Series*, of which we shall treat more fully hereafter, when the Learner has seen more of the Nature and Use of this Sort of Arithmetic.

CHAP.

CHAP. VIII.

The RULES *for ordering ſimple* EQUATIONS.

210. AN EQUATION *is the Expreſſion of Equality between two Quantities*; as $\frac{3}{4}x = b$, or $3x = 4b$, or $x = \frac{4}{3}b$. Now the Uſe of an Equation is to give the Value of an unknown Quantity x on one Side, in others which are known on the other Side; for though $\frac{3}{4}x = b = 9$, it will not ſo eaſily appear what x is, till you get it by it's ſelf on one Side of the Equation, by reaſoning thus; if $\frac{3}{4}x = 9$, then $3x = 4 \times 9 = 36$ (by Axiom 4.) and ſo $x = \frac{36}{3} = 12$, (by Axiom 5.) and is therefore known.

211. And it is the whole Deſign of this Analytic Science to expreſs the Parts and Conditions of any Problem or Queſtion in Symbols, and to ſupply Rules for the due ordering and forming ſuch a Proceſs as ſhall at laſt produce an Equation, with the unknown Quantity on one Side, and thoſe that are known on the other. The Rules for this Purpoſe are as follow.

RULE I.

212. *Any Quantity may be tranſpoſed from one Side of an Equation to the other, by changing its Sign.* For this is nothing more than to add the ſame Quantity on both Sides with a different Sign. (See Inſt. 199.)

Thus, ſuppoſe $x + 8 = 53$.
Then by Tranſpoſition, $x = 53 - 8 = 45$.
Again, let $5x - 4b = 4x + 10$
By Tranſpoſition, $5x - 4x = x = 10 + 4b$.
If $2x + a = x + b$
Then $2x - x = x = b - a$.

RULE II.

213. *Any Quantity by which the unknown Quantity is multiplied may be taken away, by dividing all the Quantities in the Equation by it.* This is evident from Inſt. 202.

Thus, if $ax = b$, then dividing both Sides by a, we have $x = \frac{b}{a}$.

Again, ſuppoſe $3x + 12 = 27$

Then, by Rule I. $3x = 27 - 12 = 15$

And, by Rule II. $x = \frac{15}{3} = 5$.

Alſo, if $ax + 2ba = 3c$

Then, Rule I. $ax = 3c - 2ba$

And by Rule II. $x = \frac{3c}{a} - 2b$.

RULE III.

214. *If the unknown Quantity be divided by any Quantity, that Quantity may be taken away by multiplying all the other Parts of the Equation by it.* (See Inſt. 201.)

Thus, if $\frac{x}{b} = b + 5$

Then ſhall $x = bb + 5b$.

If $\frac{x}{5} + 4 = 10$

Then $x + 20 = 50$

And ſo $x = 50 - 20 = 30$, by Rule I.

If $\frac{4x}{3} + 24 = 2x + 6$

Then $4x + 72 = 6x + 18$

And (Rule I.) $72 - 18 = 6x - 4x = 2x = 54$

Therefore $x = \frac{54}{2} = 27$.

RULE IV.

215. *If the unknown Quantity be concerned in Fractions, and there be more ſuch Fractions than one, they may be reduced to a common Denominator, by which, if you multiply all the Terms, the unknown Quantity will be diſengaged as before.*

Thus, let $\frac{x}{5} + \frac{x}{3} = x - 7$

Then $\frac{3x + 5x}{15} = \frac{8x}{15} = x - 7$

Conſequently, $8x = 15x - 105$

Whence, $7x = 105$, and $x = \frac{105}{7} = 15$.

RULE

RULE V.

216. *If the unknown Quantity be contained in a Surd Root, it will be equated with the known Quantities by involving both Sides of the Equation, to the proper Power.*

Thus, ſuppoſe . . . . $\sqrt{4x + 16} = 12$

Then, by ſquaring each Side, $4x + 16 = 144$

By Tranſpoſition . . . . $4x = 144 - 16 = 128$

Therefore, ... $x = \frac{128}{4} = 32.$

Again, if . . . $\sqrt{ax + b^2} - c = d$

Then, ... $\sqrt{ax + b^2} = d + c$

And, by ſquaring $ax + b^2 = d^2 + 2dc + c^2$

Whence . . . . $x = \frac{d^2 + 2dc + c^2 - b^2}{a}.$

Laſtly, if . . . . $\sqrt[3]{a^2x - b^2x} = a$

Then, ... $a^2x - b^2x = a^3$

And, ... $x = \frac{a^3}{a^2 - b^2}.$

RULE VI.

217. *If that Side of the Equation which contains the unknown Quantity be a compleat Square, Cube, or other Power; then will the unknown Quantity be equated with known Ones by Extraction of the proper Root.*

For Example, let $x^2 = 144$

Then, ... $x = \sqrt{144} = 12.$

Again, let $x^2 + 6x + 9 = 20$

Then, . . $x + 3 = \sqrt{20}$

And, . . $x = \sqrt{20} - 3.$

If we have $x^2 + ax + \frac{a^2}{4} - b^2$

 Then,

Then, $x + \frac{a}{2} = \pm b^2$*

And, $x = \pm b^2 - \frac{a}{2}$.

RULE VII.

218. *If the unknown Quantity be contained in the Terms of an Analogy, it may be had by multiplying Extremes and Means together for an Equation.* (See Inst. 104, 105.)

Thus, suppose $12 - x : \frac{x}{2} :: 4 : 1$

Then, ... $12 - x = 2x$, and $12 = 3x$, and $x = 4$.

Or, if $20 - x : x :: 7 : 3$

Then, $60 - 3x = 7x$; or $10x = 60$, and $x = 6$.

RULE VIII.

219. *If any Quantity be found on both Sides the Equation, or multiplied into all the Terms, or dividing them all, it may be struck out of the Equation.* Thus,

If $3x + b = a + b$; then $3x = a$, and $x = \frac{a}{3}$.

Again, if $3ax + 5ab = 8ac$; then $3x + 5b = 8c$, and $x = \frac{8c - 5b}{3}$.

If $\frac{2x}{3} + \frac{8}{3} = \frac{16}{3}$; then $2x + 8 = 16$, and $x = 4$.

RULE IX.

220. *Instead of any Quantity in an Equation you may substitute another of equal Value.*

Thus, if $3x + y = 24$; and $y = 9$;

Then, $3x + 9 = 24$; and $x = \frac{24 - 9}{3} = 5$.

The

* The Reason why both the Signs + and − are here placed before b^2 will be shewn a little farther on.

The Rules hitherto delivered relate to Equations which contain but one unknown Quantity; but when one or more are concerned in the Queſtion, you proceed by the following Rules.

RULE X.

221. *If there be two unknown Quantities* x *and* y, *there muſt be two Equations ariſing from the Conditions of the Queſtion; from which a Value of either* x *or* y *muſt be found in each Equation, and putting theſe Values equal to each other, a new Equation will ariſe, involving only one unknown Quantity, which is then found by the foregoing Rules.*

Example I.

Suppoſe the Sum of any two Quantities $\left.\right\}$ $x + y = s$

And their Difference $x - y = d$

Then we have . . $\begin{cases} x = s - y \\ x = d + y \end{cases}$

Therefore . . . $d + y = s - y$

Conſequently . . . $2y = s - d$

Hence . . . . . $y = \dfrac{s - d}{2}$

And by adding the two Equations $x = \dfrac{s + d}{2}$.

Example II.

Suppoſe $x + y = s$

And let $x : y :: a : b$

Then . . . $bx = ay$

And . . . . $x = \dfrac{ay}{b}$

But alſo . . . $x = s - y$.

Therefore . . $s - y = \dfrac{ay}{b}$

Hence . . . . $sb - by = ay$

And . . . . . $sb = ay + by$

Conſequently $y = \dfrac{sb}{a + b}$

And,

And, lastly $x = \frac{av}{b} = \frac{sa}{a+b}$.

EXAMPLE III.

$$\begin{aligned} \text{Let } & x + y = s \\ \text{And } & x^2 - y^2 = d \end{aligned}$$

$$\text{Then } x = s - y$$

$$\text{The Square of which, } x^2 = s^2 - 2sy + y^2$$

$$\text{Also } x^2 = d + y^2$$

$$\text{Therefore } d + y^2 = s^2 - 2sy + y^2$$

$$\text{Hence } d = s^2 - 2sy$$

$$\text{And } 2sy = s^2 - d$$

$$\text{So } y = \frac{s^2 - d}{2s}$$

$$\text{And } x = \frac{s^2 + d}{2s}.$$

RULE XI.

222. *When there are three unknown Quantities* x, y, *and* z, *there must be three Equations given, by which they may be determined; by comparing and equating these, two Equations may be obtained involving only two unknown Quantities, which are then known by the foregoing Rule.*

EXAMPLE.

$$\text{Suppose}\left\{\begin{aligned} x + y + z &= 12 \\ x + 2y + 3z &= 20 \\ \frac{x}{3} + \frac{y}{2} + 7 &= 6 \end{aligned}\right.$$

$$\text{Then we have}\left\{\begin{aligned} &1.\ x = 12 - y - z. \\ &2.\ x = 20 - 2y - 3z. \\ &3.\ x = 18 - \tfrac{3}{2}y - 3z. \end{aligned}\right.$$

$$\begin{matrix}\text{From whence we have} \\ \text{these two Equations}\end{matrix}\left\{\begin{aligned} 12 - y - z &= 20 - 2y - 3z \\ 12 - y - z &= 18 - \tfrac{3}{2}y - 3z. \end{aligned}\right.$$

There-

Therefore we have (by Rule X.) $y = 4$, $z = 2$, and consequently $x = 6$.

223. Sometimes the Equations are such, that the same Quantities in different Equations may have contrary Signs, and destroy each other; or be otherwise affected, so as to shorten the common Process very much.

Thus, suppose $\begin{cases} x + y + z = 26 \\ x - y = 4 \\ x - z = 6 \end{cases}$

Then by Addition only, $3x = 36$

Hence $x = 12$; $y = x - 4 = 8$; and $z = x - 6 = 6$.

RULE XII.

224. *If, in a general Way, the unknown Quantities* x *and* y *have Coefficients, and the Value of the Equations are expressed in Symbols, as thus,* $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ *it will always be* $y = \dfrac{af - dc}{ae - db}$; *and* $x = \dfrac{ce - bf}{ae - db}$; *which are general Theorems for the Values of* x *and* y *in all Cases.*

For in the first Equation, $ax = c - by$, and $x = \dfrac{c - by}{a}$.

And in the Second, $dx = f - ey$, and $x = \dfrac{f - ey}{d}$.

Therefore $\dfrac{c - by}{a} = \dfrac{f - ey}{d}$, and $cd - dby = af - aey$.

Whence $aey - dby = af - cd$.

Consequently $y = \dfrac{af - dc}{ae - db}$; and $x = \dfrac{ce - bf}{ae - db}$.

Example in Numbers. Suppose $\begin{cases} 5x + 7y = 100 \\ 3x + 8y = 80 \end{cases}$

Then $y = \dfrac{5 \times 80 - 3 \times 100}{5 \times 8 - 3 \times 7} = \dfrac{100}{19} = 5\dfrac{5}{19}$, and $x = \dfrac{240}{19} = 12\dfrac{12}{19}$.

225. After the same Manner, you may raise general Theorems when there are three unknown Quantities, x, y, and z, and

and three Equations given; but more of this in another Place. To conclude, we see by the above Rules, *That when there are as many simple Equations given as Quantities required, the Question is limited, and the Quantities may be discovered and determined by those Rules.*

226. But, *if there are more Quantities required, than Equations given, the Question is not limited to determinate Quantities. And, if there are more Equations given, than Quantities required, it may be impossible to find Quantities that may answer the Conditions of the Question, because some of them may be inconsistent with others.*

CHAP. IX.

A Collection of such Questions as produce SIMPLE EQUATIONS.

227. THE Order of our Institutions now brings us to consider those Questions which produce *Simple Equations*; and these will be found not only proper Exercises for the Learner, but the first Example of the Use of this excellent Art; and here we shall follow the Method invented by Mr. *Ward*, on Account of its Perspicuity and Ease, *viz.* that of numbering and registering each Step of the Process, as in the following *Solutions.*

QUESTION I.

228. If the Sum of two Numbers be 20, and their Difference 12; what are those Numbers?

First let	1	$x =$ the Greater Number
And	2	$y =$ the Lesser.
Then	3	$x + y = 20$ } per Question.
And	4	$x - y = 12$ }
Then (212)	5	$x = 20 - y$,
And	6	$x = 12 + y$;
Consequently	7	$20 - y = 12 + y$.
Then by	8	$20 - 12 = 2y = 8$; hence $y = 4$,
And *per* Question	9	$x + 4 = 20$,
Therefore . . .	10	$x = 20 - 4 = 16$; See (Inst. 221.)

where you observe two general Theorems for any Question of this Kind.

229.

QUESTION II.

229. Suppose there are two Numbers whose Sum is 32, and their Ratio as 5 to 3. Query those Numbers?

Let the Numbers be	1	$x+y=32$ } The Data.
And	2	$x:y::5:3$ }
Then (138)	3	$3x=5y$
Divide by 3, . . .	4	$x=\frac{5}{3}y$
Also	5	$x=32-y$
Therefore . . .	6	$32-y=\frac{5}{3}y$
Multiply by 3, . .	7	$96-3y=5y$
And	8	$96=8y$
Wherefore . .	9	$y=\frac{96}{8}=12.$
But by 5th . .	10	$x=32-y=32-12=20.$

The two Numbers therefore are 20 and 12, for $20+12=32$; and $20:12::5:3$, as required.

QUESTION III.

230. A Person being asked how old he was, replied, that $\frac{3}{4}$ of his Age multiplied by $\frac{1}{12}$ of his Age, gives a Product equal to his Age. Query what was his Age?

Put	1	$x=$ to his Age.
Then *per* Question	2	$\frac{3x}{4}\times\frac{x}{12}=x$
That is	3	$\frac{3x^2}{48}=x$
Whence (by 214)	4	$3x^2=48x$
That is, (by 219)	5	$3x=48$
Consequently (by 213)	6	$x=16=$ the Age required.

QUESTION IV.

231. A Person has six Sons, each of which is 4 Years older than the next younger Brother; and the Eldest is 3 Times as old as the Youngest. What are their several Ages?

Let their several Ages be . . .	1	$x,\ x+4,\ x+8,\ x+12,\ x+16$, and $x+20.$
But *per* Question	2	$3x=x+20.$
Then, transposing,	3	$3x-x=20.$
That is	4	$2x=20.$
Therefore . . .	5	$x=10.$

So their Ages are, 10, 14, 18, 22, 26, and 30, as required.

Question V.

232. A Privateer running at the Rate of 10 Miles an Hour, discovers a Ship 18 Miles off making Way at the Rate of 8 Miles an Hour: It is demanded how many Miles the Ship can run before she be overtaken?

Put	1	$x =$ Miles the Ship runs.
And	2	$y =$ Miles the Privateer runs.
Then by Supposition	3	$x : y :: 8 : 10$;
Therefore . .	4	$10x = 8y$, and $x = \frac{4}{5}y$.
But also . . .	5	$x = y - 18$;
Therefore . .	6	$\frac{4y}{5} = y - 18$,
Whence . . .	7	$4y = 5y - 90$
Consequently . . .	8	$y = 90$; and $x = y - 18 = 72$.

To find the Time; you say, as 8 Miles : 1 Hour :: 72 Miles : 9 Hours.

Question VI.

233. It is required to divide the Number 50 into two such Parts, that $\frac{3}{4}$ of one Part being added to $\frac{5}{6}$ of the other, may make 40.

Put $x\,y$ for the two Parts.

Then	1	$\frac{3x}{4} + \frac{5y}{6} = 40$.
Multiply by 4,	2	$3x + \frac{20y}{6} = 160$
Multiply by 6,	3	$18x + 20y = 960$
Therefore . .	4	$20y = 960 - 18x$
Whence . . .	5	$y = \frac{960 - 18x}{20} = 50 - x$
Hence . . .	6	$960 - 18x = 1000 - 20x$
Then . . .	7	$1000 - 960 = 2x = 40$
Consequently . . .	8	$x = 20$; and therefore $y = 30$.

Question VII.

234. Two Persons, *A* and *B*, were talking of their Money: says *A* to *B*, lend me five Shillings of your Money, and I shall have just as much as you will have left: Says *B* to *A*, rather lend me five Shillings of your Money, and I shall then have just three Times as much as you will have left: How much Money had each?

Put

Put $x = A$'s Money, and $y = B$'s.

Then, if *A* borrows,	1	$x + 5 = y - 5$
So that	2	$y = x + 10$
If *B* borrows, then	3	$y + 5 = 3 \times \overline{x - 5} = 3x - 15$
Therefore (*A*) . .	4	$x + 10 + 5 = x + 15 = 3x - 15$
And so	5	$x + 30 = 3x$;
Consequently	6	$2x = 30$, and $x = 15s$. *A*'s Money.
And	7	$x + 10 = y = 25s$. *B*'s Money.

Question VIII.

235. A Person has three Debtors, *A*, *B* and *C*, whose particular Debts he has forgot; but thus much he could remember from his Accounts, that *A*'s and *B*'s Debts together amounted to 60 Pounds; *A*'s and *C*'s to 80 Pounds; and *B*'s and *C*'s to 92 Pounds: I demand the Particulars?

The Debts let be represented by x, y, and z.

Then	1	$x + y = 60$
	2	$x + z = 80$ } *per* Question.
	3	$y + z = 92$
Then the 3—2d Step	4	$y - z = 92 - 80 = 12$
But 1st Step . . .	5	$y + z = 60$
Therefore	6	$2y = 60 + 12 = 72$; and $y = 36$.
Hence . . .	7	$x + y = x + 36 = 60$; and $x = 60 - 36 = 24$.
And lastly	8	$z = 92 - y = 92 - 36 = 56$.

So *A*'s Debt is 24*l.* *B*'s 36*l.* and *C*'s 56*l.*

Question IX.

236. There is a certain Fish whose Head is 9 Inches; the Tail is as long as the Head and half the Back; and the Back is as long as both the Head and Tail together. I demand the Length of the Back, and of the Tail?

Put	1	$x =$ Length of the Back;
Then since	2	$9 =$ Length of the Head,
We have . . .	3	$x - 9 =$ the Length of the Tail.
But, by Supposition	4	$x - 9 = \frac{x}{2} + 9$;
Therefore . .	5	$2x - 18 = x + 18$;
And so . . .	6	$2x - x - 18 = 18$.
That is . . .	7	$x - 18 = 18$, or $x = 36 =$ L. of the Back.
And	8	$x - 9 = 27 =$ Length of the Tail.

So the whole Length of the Fish was $9 + 36 + 27 = 72$ Inches, and $27 = 9 + \frac{36}{2}$, and $36 = 9 + 27$.

Question X.

237. One has a Leafe for 99 Years; and being afked how much of it was already expired, anfwered, that two Thirds of the Time paft was equal to four Fifths of the Time to come: Quere the Times paft, and to come?

Put	1	$x =$ Time paft.
Then	2	$99 - x =$ Time to come.
But	3	$\frac{2x}{3} = \frac{4}{5}$ of $99 - x = \frac{396 - 4x}{5}$;
Therefore	4	$2x = \frac{1188 - 12x}{5}$
Confequently . .	5	$10x = 1188 - 12x$
Hence	6	$10x + 12x = 22x = 1188$.
Therefore . .	7	$x = \frac{1188}{22} = 54$ Years, the Time paft.
And	8	$99 - x = 45$, the Years to come.

Thus $\frac{2}{3}$ of 54 is 36, and $\frac{4}{5}$ of 45 is 36, as required.

Question XI.

238. A Gentleman diftributing Money among fome poor People, found he wanted 10*s.* to be able to give 5*s.* to each, therefore he gives each 4*s.* only, and finds he has 5*s.* left. Query the Number of Shillings and poor People?

Let $x =$ Number of People, and $y =$ Number of Shillings.

Then by Suppofition . . .	1	$5x = y + 10$
	2	$4x = y - 5$
Then	3	$5x - 10 = y$
And	4	$4x + 5 = y$,
Therefore . . .	5	$5x - 10 = 4x + 5$
Confequently . . .	6	$5x - 4x = x = 15 =$ the poor People.
And therefore . .	7	$y = 4x + 5 = 65 =$ the Shillings.

Question XII.

239. Suppofe the Diftance between *London* and *Edinburgh* to be 360 Miles, and that a Courier fets out from *Edinburgh* running at the Rate of 10 Miles an Hour: Another fets out at the fame Time from *London*, and runs the Rate of 8 Miles an Hour. It is required to know where they will meet?

Suppofe the Courier from *Edinburgh* runs x Miles, and the other y Miles before they meet.

Then

Then . . .	1	$x + y = 360$ } by Supposition
	2	$x : y :: 5 : 4$ }
Therefore . . .	3	$4x = 5y$; and $x = \frac{5}{4}y$.
Alſo	4	$x = 360 - y = \frac{5}{4}y$
Hence	5	$\frac{5}{4}y + y = 360$.
Therefore . . .	6	$9y = 1440$.
Conſequently	7	$y = 160$.
	8	$x = 360 - y = 200$.

Question XIII.

240. One ſets out from a certain Place and travels at the Rate of 7 Miles in 5 Hours, and 8 Hours after another ſets out from the ſame Place and travels the ſame Road at the Rate of 5 Miles in 3 Hours. How long and how far muſt the Firſt travel before he is overtaken by the Second.

Put . . .	1	$x =$ Hours the 1ſt travelled.
Then . .	2	$x - 8 =$ Hours the 2d travelled.
Now ſay,	3	$5 : 7 :: x : \frac{7}{5}x =$ Whole Diſtance.
Alſo . .	4	$3 : 5 :: x - 8 : \frac{5x - 40}{3} =$ Whole Diſtance.
Therefore	5	$\frac{5x - 40}{3} = \frac{7}{5}x$.
Whence	6	$25x - 200 = 21x$.
Therefore	7	$25x - 21x = 4x = 200$
Hence . .	8	$x = 50$; and $x - 8 = 42$.
And ſo	9	$\frac{7}{5}x = 70$ Miles they both travelled.

241. Theſe Examples may ſuffice for a mixt or numerical Analyſis; but Queſtions ſolved in a general Way by Symbols only, are much more uſeful ſince they then become general Canons or Theorems, and are applicable to every particular Caſe; whereas the other Way in Numbers is confined to one Caſe only; therefore we ſhall proceed to give a few Examples of general Solutions. Thus,

Question XIV.

242. Two Couriers, A and B, ſet out from two Places diſtant from each other d Miles; of which the foremoſt A travels p Miles in q Hours, and B travels r Miles in s Hours; he is ſuppoſed to walk faſteſt, and the ſame Way with A. It is required

to

to know how many Miles each must travel before B can overtake A?

Put . .	1	$x =$ Miles A travelled
	2	$y =$ Miles B travelled
Then also	3	$x = y - d$
And we have	4	$q : p :: 1 : \frac{p}{q} =$ Miles *per* Hours by A
Also . . .	5	$s : r :: 1 : \frac{r}{s} =$ Miles B walks *per* Hour
Then we have	6	$x : y :: \frac{p}{q} : \frac{r}{s}$
Hence . .	7	$\frac{xr}{s} = \frac{py}{q}$
Therefore	8	$qrx = psy$
Whence . . .	9	$\frac{psy}{qr} = x = y - d$
And so	10	$psy = qry - qrd$
And . . .	11	$psy + qrd = qry$
Whence . . .	12	$qrd = qry - psy$
Therefore	13	$\frac{qrd}{qr - ps} = y =$ Miles B travelled
And lastly	14	$x = y - d =$ Miles travelled by A.

N. B. If $p = 8$, $q = 1$; $r = 10$, $s = 1$; and $d = 18$; then $y = 90$, $x = 72$, as in Question 11. Thus A is the Ship, and B the Privateer that pursued her. But more generally yet;

Question XV.

243. Let A be a moveable Body, which in Time f can describe the Space c; and let B be another moving Body, which in the Time g can describe the Space d; let the Distance from which they begin to move, be called a and b, the Difference of the Time.

In the Solution of this Problem, there will be two Cases, the first is, when both the Bodies, A and B, tend towards the same Parts, or move the same Way. The second Case is, when they tend towards contrary Parts, or meet each other. In both Cases we suppose the Body A to be farthest from the Place where they come together.

CASE

CASE I.

244. Let that Distance of A be called x, from which take the Interval e, and there will remain $x-e$ for the Distance of B, from the said Place where they come together. And since A passeth over the Space c in the Time of f, it will pass over the Space x in the Time of $\frac{fx}{c}$, for it is $c:f::x:\frac{fx}{c}$.

Also since B passeth over the Space d in the Time g, it will pass over the Space $x-e$ in the Time $\frac{gx-ge}{d}$; since $d:g::x-e:\frac{gx-ge}{d}$.

Now since the Difference of those Times is supposed to be h, that they may become equal; we must add h to the shortest Time, that is, (since A is supposed to move first) to the shortest Time of B, viz. to $\frac{gx-ge}{d}$; and then we have the Time equalled, viz. $\frac{gx-ge}{d}+h=\frac{fx}{c}$; and which reduced, gives $x=\frac{cge-cdh}{cg-df}$. But if the nearest Body B begin to move first, h must be added to the shorter Time of A, viz. $\frac{fx}{c}$ and we have $\frac{fx}{c}+h=\frac{gx-ge}{d}$; which reduced gives $\frac{cge+cdh}{cg-fd}=x$.

CASE II.

245. If the Bodies, A and B meet; then if $x=$ Distance of the remotest Body A from the Place where they meet, as before, we have $e-x=$ Distance of B from the same Spot. And the Times in which they pass over those Spaces will be $\frac{fx}{c}$, and $\frac{ge-gx}{d}$, found as before. Then if A moves first, we have $\frac{ge-cx}{d}+h=\frac{fx}{c}$ and this Equation gives $\frac{cge+cdh}{cg+df}=x$.

But

But if B move first, we have $\frac{fx}{c} + h = \frac{ge - gx}{d}$; whence we get $\frac{cge - cgh}{cg + df} = x$.

Example I.

246. The Sun describes each Day a Degree in the Ecliptic, and the Moon Thirteen; and at a certain Time, when the Sun is in the Beginning of *Cancer*, suppose the Moon, three Days after, to be in the Beginning of *Aries*. It is required to know in what Part of the Ecliptic the next new Moon will happen?

In order to solve this Problem, we must consider that A represents the Moon, and B represents the Sun, and that they both tend towards the same Part; and lastly, that the Motion of A is shorter than that of B; therefore the Theorem $\frac{cge + cdh}{cg - df} = x$ must be used, in which the Values of the several Letters are known. Thus,

$c = 13$ Degrees, the Moon's Motion *per* Day.
$f = 1$ Day.
$d = 1$ Degree, the Sun's Motion *per* Day.
$g = 1$ Day.
$e = 90$ Degrees, the Distance of the Sun and Moon.
$h = 3$ Days, the Difference in Time.

Hence the above Theorem is expressed in Numbers thus,

$$\frac{cge + cdh}{cg - df} = \frac{13 \times 1 \times 90 + 13 \times 1 \times 3}{13 \times 1 - 1 \times 1} = \frac{1209}{12} =$$

$100\frac{3}{4}$, that is, $100\frac{3}{4}$ Degrees from the Beginning of *Aries*, so that the Conjunction happens in $10\frac{3}{4}$ Degrees of *Cancer*.

After the same Manner, A and B may represent the two Hands of a Clock, either setting out from the same Place at the same Time, or from different Places at different Times.

Again, by Theorem $\frac{cge + cdh}{cg - df} = x$, A and B may represent two Persons setting out from two different Places, in order to meet, either at the same, or different Times, as in the former Question, for *London* and *Edinburgh*, and the Distance travelled by each will be found by the Theorem, the same as was there determined. But these Things we shall leave as a proper Exercise for the young Learner.

Chap. X.

Questions analytically solved, to illustrate the Nature *and* Reason *of the several* Rules *of* Arithmetic, *in Vulgar Computation.*

247. We shall here give the Solution of such Questions, in an Algebraic Way, as will shew the *Rationale* of the Methods used in the several Operations of *Vulgar Arithmetic*, such as the Double Rule of Three, Rules of Fellowship, Alligation, Simple Interest, False Position, *&c.* as our principal View in all these Things, is rather to make an intelligent than practical Artist, since many are ready enough at Practice, at the same Time they know little or nothing of the Reason of what they do.

I. Double Rule *of* Three.

248. In all Questions of this Kind, there are three principal Terms which contain the Supposition, and three others in which the Question lies; the three first are as follows:

A, the efficient Cause that produces an Effect;

T, the Time in which that Effect is produced;

E, the Effect produced in the given Time.

The other three Terms, which move the Question, are of the same Kind, and therefore to be denoted by the same Letters a, t, e, in small Characters; hence, in order to raise Theorems for stating Questions in this Rule, we must reason as follows:

249. In equal Times, the Effects produced will always be proportioned to the efficient Causes: This is a self-evident Axiom, or indisputable Truth. Therefore, to express this in Symbols, we must put $T = t$, and then we have this Analogy, $E : e :: A : a$; therefore $Ae = Ea$.

250. Another Axiom is, that when the efficient Causes are equal, the Effects which they produce will always be proportioned to the Times that are employed in producing them. That is in Symbols, if $A = a$, then $E : e :: T : t$, and therefore $Et = eT$.

But if B move first, we have $\frac{fx}{c} + h = \frac{ge - gx}{d}$; whence we get $\frac{cge - cgh}{cg + df} = x$.

Example I.

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CHAP.

CHAP. X.

Questions analytically solved, to illustrate the NATURE *and* REASON *of the several* RULES *of* ARITHMETIC, *in Vulgar Computation.*

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E, the Effect produced in the given Time.

The other three Terms, which move the Question, are of the same Kind, and therefore to be denoted by the same Letters a, t, e, in small Characters; hence, in order to raise Theorems for stating Questions in this Rule, we must reason as follows:

249. In equal Times, the Effects produced will always be proportioned to the efficient Causes: This is a self-evident Axiom, or indisputable Truth. Therefore, to express this in Symbols, we must put $T = t$, and then we have this Analogy, $E : e :: A : a$; therefore $Ae = Ea$.

250. Another Axiom is, that when the efficient Causes are equal, the Effects which they produce will always be proportioned to the Times that are employed in producing them. That is in Symbols, if $A = a$, then $E : e :: T : t$, and therefore $Et = eT$.

251. But when neither the Times nor the efficient Causes are equal, the Effects produced will be greater in Proportion to them both conjointly, that is, $E : e :: AT : at$, and therefore $atE = ATe$, from which general Theorem, if any five of those Quantities are given, the Sixth may be found without any regard to the Proportions being *Direct* or *Inverse*. And the three Terms of the Question may be expressed by the three following Theorems, *viz.* Theo. I. $a = \frac{ATe}{Et}$. Theo. II. $e = \frac{atE}{AT}$. Theo. III. $t = \frac{eAT}{Ea}$.

Example I.

252. Suppose 100*l.* in 12 Months, gain 4*l.* Interest, what Principal or Sum must be put out to gain 132*l.* in 66 Months? To answer this Question, we have in Theorem I. the Value of the Symbols as follows:

$A = 100$, $T = 12$, $E = 4$, $t = 66$, $e = 132$, to find a, and the Theorem will stand in Numbers thus, $\frac{100 \times 12 \times 132}{4 \times 66} = a = 600l.$ the Answer.

Example II.

If 2 Men in 3 Days will earn 4 Shillings, how much will 5 Men earn in 6 Days?

In this Case, $A = 2$, $E = 4$, $T = 3$, and $a = 5$, $t = 6$. To find e, for which Purpose we must make use of Theorem II. $\frac{atE}{AT} = e$, that is in Numbers $\frac{5 \times 6 \times 4}{2 \times 3} = 20$ Shillings, the Answer.

Example III.

If for the Carriage of 300 Weight 40 Miles, I must pay 7 Shillings and 6 Pence, how far may I carry 500 Weight for 18 Shillings and 9 Pence?

In this Question we have given $A = 300$, $E = 90$ Pence, and $T = 40$, $a = 500$, $e = 225$ Pence, to find t?

N. B. In Questions of this Sort, the Letter T, or t, may denote, not only *Time*, but also *Distance*, or any other Mode, or Way by which any Cause can produce its Effect, the Solution there-

therefore of this Queſtion, is by Theorem III. *viz.* $\frac{eAT}{Ea} = t$, which in Numbers, ſtands thus, $\frac{225 \times 30 \times 40}{90 \times 500} = 60$ Miles.

The Rule *of* Fellowship.

253. Suppoſe $\begin{cases} A, B, C, D, \&c. \text{ Merchants trading in Company.} \\ a, b, c, d, \&c. \text{ Each Man's Money in Stock.} \\ w, x, y, z, \&c. \text{ The Time it is employed by each.} \end{cases}$

Now ſince each Perſon's Share, of the Loſs or Gain, muſt be proportioned to the Money he advances, and alſo to the Time it is employed in Trade, therefore his Share will be as the Product of both multiplied together; that is to ſay, $aw = A$'s Share, $bx = B$'s Share, and ſo on.

Therefore let $\begin{cases} aw + bx + cy + dz, \&c. = S, \text{ the Sum of the Products of the Times and Stocks.} \\ G = \text{the whole Gain, Loſs, \&c. by Trading.} \end{cases}$

Then it is evident as S : G : : each of the above Products : Merchant's Share of Loſs or Gain reſpectively: That is, in Symbols as follows:

$$\left.\begin{array}{l} \text{Theorem I. } S : G :: aw : \frac{G}{S} \times aw = A\text{'s Gain.} \\ \text{Theorem II. } S : G :: bx : \frac{G}{S} \times bx = B\text{'s Gain.} \\ \text{Theorem III. } S : G :: cy : \frac{G}{S} \times cy = C\text{'s Gain.} \\ \text{Theorem IV. } S : G :: dz : \frac{G}{S} \times dz = D\text{'s Gain.} \end{array}\right\} \&c.$$

Example.

254. Three Merchants *A*, *B*, and *C*, enter into Partnerſhip, thus;

A puts in 60*l.* for 8 Months, *B* puts in 75*l.* for 12 Months, and *C* puts in 80*l.* for 9 Months, with this joint Stock they Traffic and Gain a 150*l.* it is required to find each Man's Share proportioned to his Stock and Time.

$$\text{First}\begin{cases} A\text{'s Stock } 60l. \times 8 = 480. \\ B\text{'s Stock } 75l. \times 12 = 900. \\ C\text{'s Stock } 80l. \times 9 = 720. \end{cases}\text{Then ſay,}$$

The Sum of the Products = 2100.

As

$$\begin{array}{llrrrl} & & l. & s. & d. & \\ \text{As } 2100 : 150 \left\{ \begin{array}{l} : : 480 : 34,\ 2857 \\ : : 900 : 64,\ 2857 \\ : : 720 : 51,\ 42857 \end{array} \right. & \begin{array}{l} = \\ = \\ = \end{array} & \begin{array}{r} 34 \\ 64 \\ 51 \end{array} & \begin{array}{r} 5 \\ 5 \\ 8 \end{array} & \begin{array}{r} 8\frac{1}{2} \\ 8\frac{1}{2} \\ 6\frac{3}{4} \end{array} & \begin{array}{l} A\text{'s Gain.} \\ B\text{'s Gain.} \\ C\text{'s Gain.} \end{array} \end{array}$$

N. B. When all the Stocks are employed an equal Time, the Theorems above will become more Simple, for then we neglect the Characters, w, x, y, z, as being each equal to Unity, in which Case those Theorems give the Rules of what is usually called Single Fellowship, which, as it is so easy, we need not here farther insist on.

The Rule *of* Alligation.

255. Alligation is a Rule for compounding or mixing several Ingredients of different Sorts together, in any Manner or Proportion.

Let the Quantities to be compounded be a, b.
Their Prices ——— ——— ——— x, y.
The Compound which they make ——— c
And its Price ——— ——— ——— p
Then we have ——— ——— $a + b = c$
And also ——— ——— $xa + yb = pc$
Whence this Analogy, as $c : xa + yb :: 1 : p$.

That is in Words, *as the Sum of the Quantities is to the Sum of the Products of each by its Price, so is any one Part of the Compound or Mixture to the Price of that Part.*

Example.

Suppose 20 Pounds of Tobacco at 9 Pence *per* Pound, were to be mixed with 12 Pound of Ditto at 14 Pence *per* Pound, what will a Pound of that Mixture be worth? Here $a = 20$, $b = 12$; $x = 9$, $y = 14$; and $a + b = c = 32$; then $ax = 180$, $by = 168$; and $ax + yb = 348$. Therefore $32 : 348 :: 1 : 10d. \frac{3}{4}$, nearly, the Answer. This is called *Alligation Medial.*

256. When the Prices or particular Rates of the several Ingredients, and the Mixture and its Price are given, to find the particular Quantities of the Ingredients, it is called *Alligation Alternate*, as being the Reverse of the foregoing.

If

If two Quantities only are concerned, then will the Equation be (as before) $xa + yb = pc$;

Then . . . . $yb = pc - xa$

And . . . . $b = \frac{pc - xa}{y}$

Also it is . . . . $a + b = c$

Therefore . . . . $b = c - a = \frac{pc - xa}{y}$

Consequently . . . . . . $cy - ya = pc - xa$

Then . . . . $cy + xa - ya = pc$

And . . . . $xa - ya = pc - yc$

And therefore . . . . . $a = \frac{p - y}{x - y} \times c$.

EXAMPLE.

257. How much Wine at 16 Pence a Pint, and of another Sort at 10 Pence a Pint, will compose a Mixture that shall be worth 12 Pence a Pint? Here $x = 16$, $y = 10$; and $p = 12$, and $c = 1$; therefore the Rule or Theorem is $a = \frac{p - y}{x - y} = \frac{12 - 10}{16 - 10} = \frac{2}{6} = \frac{1}{3}$; therefore $\frac{1}{3}$ of a Pint of the first, and $\frac{2}{3}$ of a Pint of the second Sort must be taken.

N. B. If $y = o$, then b will represent Water, whose Price is Nothing, and then $a = \frac{p}{x} = \frac{12}{16} = \frac{3}{4}$ of a Pint.

258. When more than two Quantities are to be compounded, and all unknown, the Question will be *unlimited*, as there can be but two Equations, this will appear by the Steps of the following Process.

Let	1	a, e, y, be three Ingredients.
And	2	r, s, t, their several Prices
Also let . . .	3	m, and p, be the Compound and its Price
Then	4	$a + e + y = m$
And	5	$ar + se + ty = pm$
Step 1st $- a$ . .	6	$e + y = m - a$
$5 - ar$ . . .	7	$se + ty = pm - ra$
$6 \times$ by t . . .	8	$te + ty = tm - ta$
$7 - 8$ . . .	9	$se - te = pm - tm - ra + ta$

There-

Therefore . .	10	$e = \frac{p - t}{s - t} \times m - \frac{r - t}{s - t} \times a.$
Hence by 6th and 9th	11	$y + \frac{pm - tm - ra + ta}{s - t} = m - a$
And reducing, . .	12	$\left\{ \begin{array}{l} sy - ty + pm - tm - ra + ta \\ = sm - sa - tm + ta. \end{array} \right.$
That is . . .	13	$sy - ty + pm - ra = sm - sa$
And	14	$sy - ty = sm - pm - sa + ra$
Consequently . . .	15	$y = \frac{r - s}{s - t} \times a - \frac{p - s}{s - t} \times m.$

Now here it is evident from the 10th Step, that a must be less than $\frac{p - t}{r - t} \times m$; and from the last Step it appears, that a must be grearer than $\frac{p - s}{r - s}$. Therefore any Value of a may be taken between these two Limits.

Example.

259. How much Tobacco at 2*s.* 8*d.* *per* Pound, and of another Sort at 20*d.* and of a third Sort at 16*d.* *per* Pound, must be taken to make a Mixture that shall contain 56 ℔, and may be sold for 22*d.* *per* ℔, without Loss or Gain?

Here $r = 32$, $s = 20$, $t = 16$; $m = 56$, and $p = 22$; also a is the Quantity worth 32*d.* *per* Pound, e that of 20*d.* and y that of 16*d.* *per* Pound, hence the greatest Limit of a, will be $\frac{p - t}{r - t} \times m = \frac{22 - 16}{32 - 16} \times 56 = 21$, that is to say, a must be less than 21. Again, the least Limit of a, will be $\frac{p - s}{r - s} \times m = \frac{22 - 20}{32 - 20} \times 56, = 9\frac{1}{3}$; therefore the least Quantity of a, must be greater than $9\frac{1}{3}$. Hence 11 Answers may be obtained for this Question, in whole Numbers, and an infinite Number in Fractions. Thus if the Values of a, be as in the first Column of the following Table, then the Values of e and y, by the 10th and 15th Steps, will be found such as are contained in the 2d and 3d Columns.

a

a	*e*	*y*	*a*	*e*	*y*	*a*	*e*	*y*
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

260. After the same Manner, if 4 Quantities or more are to be mixed together, you may proceed in the same Manner to find the Limits of the 1st and 2d, because all the Quantities except the two last will admit of different Values, the Process will be large and troublesome; but if the Learner will pursue it, the Method is the same as before, and the Reason and Truth of the following Theorems will then appear, which I investigated many Years since for perfecting this Part of Arithmetic.

$$\text{Let}\left\{\begin{array}{l}a=\\ b=\\ c=\\ d=\\ x=\end{array}\right\}\text{Diff. between}\left\{\begin{array}{l}\text{the first Rate and the last}\\ \text{the 2d and last}\\ \text{the 3d and last}\\ \text{the 4th and last}\\ \text{the Mean Rate and the last.}\end{array}\right.$$

$$\text{Also let}\left\{\begin{array}{l}a=\\ b=\\ c=\\ d=\\ z=\end{array}\right\}\text{Diff. bet.}\left\{\begin{array}{l}\text{the First}\\ \text{the 2d}\\ \text{the 3d}\\ \text{the 4th}\\ \text{the Mean}\end{array}\right\}\text{and the last Rate but one.}$$

$$\text{And}\left\{\begin{array}{l}p=\\ q=\\ n=\\ S=\text{To the Sum of all the Ingredients, } A+B+C\end{array}\right.\text{Diff. between}\left\{\begin{array}{l}\text{Mean Rate and the second}\\ \text{the first Rate and second}\\ \text{the last but one and the last}\end{array}\right.$$

$+ D + E + F$, &c.

261. Then if there be three Quantities to be compounded $A + B + C = S$, the Theorems are as follow. Theo. I. $\frac{Sx}{a} = A$, greatest. Theor. II. $\frac{Sp}{q} = A$, least. Theor. III. $\frac{Sx - Aa}{n} = B$. Theor. IV. $S - A - B = C$.

262. But if four Quantities are to be mixed, *viz.* A, B, C, D, the first is found as before; the Theorems for B and C, are as follows.

Theor.

Theor. V. $\frac{Sx - Aa}{b} = B$, greatest. Theor. VI. $\frac{Sz - Aa}{b} = B$, least. Theor. VII. $\frac{Sx - Aa - Bb}{n} = C$; and then D is known of Course.

263. If there be five Ingredients, A, B, C, D, E, you have the greatest and least Values of A and B as before; and for C and D as below.

Theor. VIII. $\frac{Sx - Aa - Bb}{c} = C$, Greatest.

Theor. IX. $\frac{Sz - Aa - Bb}{c} = C$, Least.

Theor. X. $\frac{Sx - Aa - Bb - Cc}{n} = D$.

264. When there are six Quantities, A, B, C, D, E, F, the greatest and least Values of the three first are as in the last; and for D and E as follows.

Theor. XI. $\frac{Sx - Aa - Bb - Cc}{d} = D$, Greatest.

Theor. XII. $\frac{Sz - Aa - Bb - Cc}{d} = D$, Least.

Theor. XIII. $\frac{Sx - Aa - Bb - Cc - Dd}{n} = E$.

And thus you proceed for any greater Number at Pleasure.

265. As to the *Rule* of False Position, as it is called, it can only be of Use to those who are unacquainted with *Algebra*; and a Person may as soon learn the *Analytic Art*, as the *Reason* and *Operations* of this Rule without it; thus for Instance, in the *Single Rule* where *one Position* (or Supposition) only is required, as in the following Question, *viz.*

266. Three Men, *A*, *B*, *C*, buy a Ship for 310*l.* 16*s.* of which *A* paid an unknown Sum; *B* paid $2\frac{1}{2}$ as much, and C $3\frac{1}{3}$ as much: How much did each Man pay?

Suppose *A* paid 48*l.* then *B* paid $48 \times 2,5 = 120$*l.* and *C* must pay $48 \times 3,\dot{3} = 160$*l.* But $48 + 120 + 160 = 328$ instead of 310,75*l.*

Say therefore, As 328 : 48 : : 310,75 : 45,4756, &c.

Then

Then *A* paid . . . . £. 45,4756
B paid $(45,4756 \times 2,5 =)$ . . . 113,689
C paid $(45,4756 \times 3,3 =)$ 151,5853

Proof is the Sum . . . . £. 310,75

267. Now the Anſwer to this Queſtion is in a moſt direct and eaſy Manner pointed out by Algebra; for let $x =$ Sum which *A* paid, then *B* paid $2,5x$, and *C* paid $3,3x$, but $x + 2,5x + 3,3x = 310,75\,l.$ Therefore $\overline{1 + 2,5 + 3,3} \times x = 6,83x$; hence $x = \frac{310,75}{6,83} = 45,4756\,l.$ the Money *A* paid; then *B*'s and *C*'s Part is found, as before. And as to the *Double Rule of Falſe Poſition*, it is ſo troubleſome in Numbers, and ſo eaſy by Algebra, that we think it unworthy of an ingenious Artiſt, and ſhall leave it to the Indolent and Ignorant only, who generally take the moſt Pains to the leaſt Purpoſe.

SIMPLE INTEREST.

268. Simple Intereſt is the Money paid for the Uſe or Loan of any Principal or Sum lent out for a given Time, at a certain Rate *per Cent. per Annum*. Our Laws regulate this Matter; and our Buſineſs is now to ſhew how the *Rules of Computation* are derived from *Theorems* inveſtigated by *Algebraic Reaſoning*. Therefore,

Let
- P = any *Principal* or *Sum* put to Intereſt,
- R = Ratio, or Rate *per Cent. per Ann.*
- t = Time of Continuance at Intereſt,
- A = Amount of the Principal and its Intereſt.

N. B. The *Ratio of the Rate* is the Simple Intereſt of 1 *l.* for one Year, at any given Rate. Thus, as $100 : 5 :: 1 : 0,05 =$ Ratio at 5 *per Cent. per Annum*; and $100 : 3\frac{1}{2} :: 1 : 0,035 =$ Ratio at $3\frac{1}{2}$ *per Cent. per Ann.*

269. Now R = the Intereſt of 1 *l.* for one Year,
2 R = the Intereſt of 1 *l.* for two Years,
3 R = the Intereſt of 1 *l.* for three Years,
And t R = the Intereſt of 1 *l.* for t Years.

But as $1\,l.$ is to its Intereſt tR, for any given Time t, in the ſame Proportion as any Sum is to its Intereſt for the ſame Time. Hence the following Analogy:

$$1\,l. : tR :: P : tPR.$$

270. But the *Principal* and *Intereſt* added together is equal to the *Amount*, that is,

$$\text{Theorem I. } tRP + P = A = \text{the Amount.}$$

$$\text{Then Theorem II. } \frac{A}{tR + 1} = P = \text{the Principal,}$$

$$\text{And Theorem III. } \frac{A - P}{tP} = R = \text{the Ratio,}$$

$$\text{Laſtly, Theorem IV. } \frac{A - P}{RP} = t = \text{the Time.}$$

271. By theſe Theorems all Queſtious about Simple Intereſt are anſwered with the utmoſt Eaſe.

Question. I.

What will $256\,l.$ $10\,s.$ amount to in three Years, one Quarter, two Months and eighteen Days, at 4 *per Cent. per Annum?*

In order to anſwer Queſtions in this Rule, the Parts of Integers, muſt be reduced to *Decimals*, eſpecially thoſe of Time, which may be readily done by obſerving,

$$\text{That one} \begin{cases} \text{Day is } \frac{1}{365} \text{ Part of a Year} = 0{,}00274 \\ \text{Month is } \frac{1}{12} \text{ of a Year} = 0{,}08333 \\ \text{Quarter is } \frac{1}{4} \text{ of a Year} = 0{,}25 \end{cases}$$

$$\text{Then the Time is} \begin{cases} 3 \text{ Years} = 3. \\ 1 \text{ Quarter} = 0{,}25 \\ 2 \text{ Months} = 0{,}16667 \\ 18 \text{ Days} = 0{,}04932 \end{cases}$$

$$\text{That is, } t = 3{,}46599$$

Then $tR = 0{,}13864$; $tRP = 35{,}561057$; and laſtly, $tRP + P = 35{,}561057 + 256{,}5 = 292{,}061$, &c. $= 292\,l.$ $1\,s.$ $2\,d.\ \frac{1}{2} = A$, the Amount required, as *per* Theor. I.

Question II.

272. What Principal or Sum put to Intereſt at 4 *per Cent.* &c. will amount to $292\,l.$ $1\,s.$ $2\,d\,\frac{1}{2}$ in 3 Years 1 Quarter, 2 Months, 18 Days?

Anſwer,

Answer, per Theorem II. $\frac{A}{tR+1} = \frac{292,061}{0,13864+1} = \frac{292,061}{1,13864}$ $= 256,5 = P = 256\,l.\ 10\,s.$

Question III.

273. At what Rate *per Cent.* Interest will 256 *l.* 10 *s.* amount to 292 *l.* 1 *s.* 2 *d.* ½ in 3 Years, 1 Quarter, 2 Months, 18 Days?

Answer, *per* Theorem III. $\frac{A-P}{tP} = \frac{292,061 - 256,5}{889,0234} = \frac{35,561}{889,0234} = 0,04 = R.$ So the Interest is 4 *per Cent.*

Question IV.

274. In what Time will 256 *l.* 10 *s.* raise a Stock or amount to 292 *l.* 1 *s.* 2 *d.* ½, at the Rate of 4 *per Cent.* Interest?

Answer, *per* Theorem IV. $\frac{A-P}{RP} = \frac{35,561}{10,26} = 3,46599$ $= t = 3$ Years, 1 Quarter, 2 Months, 18 Days.

275. If it be considered, that t stands for any Time indifferently, as well a Part of a Year as a whole Year, or any Number of Years, it must appear very strange for any one to imagine that, if 100 *l.* *in 12 Months gain 4 l. Interest, it will not gain 2 l. in 6 Months.* Yet this Mr. *Ward* affirmed (in his *Scholium*, p. 248) contrary to the Reason of Things, and to his first or *fundamental Theorem*, viz. $tRP + P = A$; for let $P = 100\,l.$ $R = 0,04$; $t = 0,5$; then $tRP + P = \overline{0,5 \times 0,04 \times 100} + 100 = 2 + 100 = 102\,l. = A$, the true Amount in six Months; and therefore 2 *l.* must be the Interest for that Time.

N. B. What farther relates to Simple and Compound Interest must be deferred 'till after we have delivered the Doctrine of *Geometrical Progression*, and *Quadratic Equations*, which we next proceed to.

CHAP.

CHAP. IX.

Of INVOLUTION.

276. BY *Involution* we mean a continual Multiplication of the ſame Quantity into itſelf; and the Products ariſing from thence, are called the *Powers* of that Quantity; and the Quantity itſelf is called the *Root*. Thus $a \times a = aa$; and $aa \times a = aaa$, &c. So $ab \times ab = aabb$; and $aabb \times ab = aaabbb$; and ſo on. But when the ſame Quantity is often repeated, it is uſual to write it but once with a Figure above to ſhew how often it is repeated in any Product or Power; as thus for aa we write a^2; for aaa, a^3; for $aaabbb$, $a^3 b^3$; and the Figures are called the *Indices*, or Exponents of thoſe Powers, as they ſhew how many Times the Root has been involved from Unity as in the Table below.

277. 1 *Unity.*
$a =$ *Root*, or 1ſt Power.
$aa = a^2$, the *Square*, or 2d Power.
$aaa = a^3$, the *Cube*, or 3d Power.
$aaaa = a^4$, the *Biquadrate*, or 4th Power.
$aaaaa = a^5$, the *Surſolid*, or 5th Power.

And ſo on, as far as you involve the Root. Note, the Index of a^1 the firſt Power, being Unity, is never wrote; and we make $a^0 = 1$, or the firſt Term of the Series, when we would expreſs the Series compleat in Terms of a; as a^0, a^1, a^2, a^3, &c.

278. Hence it is evident, that theſe Powers are a Series of Terms in *Geometrical Proportion*, and their *Indices* the *Logarithms* of the Terms; the very ſame as we obſerved before of a like Series of Geometrical Terms in *Numbers* (130, 131, 132, &c.) Whence alſo it follows, that theſe Powers are *multiplied* and *divided* by *adding* and *ſubtracting their Indices* (132, 133.) Thus $a^2 \times a^3 = a^{2+3} = a^5$; and $a^2 b^2 \times a^3 b^3 = a^5 b^5$; and ſo in Diviſion $\frac{a^5}{a^2} = a^{5-2} = a^3$, and $\frac{a^5 b^5}{a^3 b^3} = a^{5-3} b^{5-3} = a^2 b^2$.

279. Hence likewiſe it appears, if a leſſer Power be divided by a greater the Index or Exponent of the Quotient muſt be ne-
gative:

gative: Thus $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, but $\frac{a^3}{a^5} = \frac{1}{a^2}$; therefore also $\frac{1}{a^2} = a^{-2}$, (see 198).

Again, it is obvious, that $\frac{a}{a} = a^{1-1} = a^0$; but, $\frac{a}{a} = \frac{1}{1} = 1$; therefore $a^0 = 1$; as before observed. In the same Manner $\frac{1}{a} = \frac{a^0}{a} = a^{0-1} = a^{-1}$; and therefore 'tis evident $aaa.\ aa.\ a.\ 1.\ \frac{1}{a}.\ \frac{1}{aa}.\ \frac{1}{aaa}$, &c. may be thus expressed, $a^3.\ a^2.\ a^1.\ a^0.\ a^{-1}.\ a^{-2}.\ a^{-3}$, &c.

280. The *negative* Powers of a, are *positive* Powers of $\frac{1}{a}$ or a^{-1}; and are *multiplied and divided* by *adding and subtracting* their Indices as before in the positive ones. Thus, $a^{-2} \times a^{-3} = \left(\frac{1}{aa} \times \frac{1}{aaa} = \frac{1}{a^5} = \right) a^{-2-3} = a^{-5}$. And $\frac{a^{-5}}{a^{-3}} = \frac{1}{a^5} \div a^3 = a^{-5+3} = a^{-2}$; because $\frac{1}{a^3}\Big) \frac{1}{a^5} \Big(= \frac{a^3}{a^5} = \frac{1}{a^2} = a^{3-5} = a^{-2}$ (279.)

281. Hence if a *positive Power* be multiplied by an equal *negative Power* of any Quantity a, the Product will be *Unity*; *thus* $a^{-3} \times a^3 = a^{3-3} = a^0 = 1$; for positive and negative Powers destroy each other, and produce only Unity, which is no Power of a at all.

282. Suppose $\frac{a^{-3}}{a^{-5}} = \frac{a^{-3}}{a^{-3} \times a^{-2}} = \frac{1}{a^{-2}}$. But also $\frac{a^{-3}}{a^{-5}}$ is $\frac{1}{a^5}\Big) \frac{1}{a^3} \Big(= \frac{a^5}{a^3} = a^{5-3} = a^2$, therefore $\frac{1}{a^{-2}} = a^2$. So that, in general, *any Quantity in the Denominator of a Fraction may be placed in the Numerator, with the Sign of the Index changed.* Thus, $\frac{x}{a^3 b} = \frac{x}{b} \times \frac{1}{a^3} = \frac{x}{b} \times a^{-3} = \frac{x a^{-3}}{b}$; and $\frac{x}{a^{-3} b} = \frac{x a^3}{b}$.

283. If

283. If (m) represent the indeterminate Index of any Power of a that is positive, as a^m; then $\frac{1}{a^m} = a^{-m}$, will express an equal negative Power of a. And $a^m \times a^{-m} = a^{m-m} = a^0 = 1$. Also, if a^n express any other Power of a, then $a^m \times a^n = a^{m+n}$ is the Product of the two Powers a^m and a^n; and $\frac{a^m}{a^n} = a^{m-n}$, is their Quotient.

184. To raise any simple Quantity to its 2d, 3d, 4th, &c. Power, is, multiply its Exponent by 2, 3, 4, &c. (as *per* 134.) thus the Square of a is $a^{1\times 2} = a^2$; the third Power or Cube is $a^{1\times 3} = a^3$; the mth Power of a is $a^{1\times m} = a^m$; also the Square of a^3 is $a^{3\times 2} = a^6$; the Cube of a^2 is $a^{2\times 3} = a^6$; and the mth Power of a^2 is $a^{2\times m} = a^{2m}$, and the nth Power of a^m is $a^{m\times n} = a^{mn}$. The Square of ab is $a^2 b^2$; the Cube is $a^3 b^3$; the mth Power $a^m b^m$.

185. If there are Signs and Coefficients, the Signs must be ordered as directed in Multiplication, and the Coefficients involved, *viz.* squared, cubed, &c. as in the Examples below.

$$
\begin{array}{rrl}
+2a & -3a & \text{First Power.} \\
+2a & -3a & \\
\hline
+4a^2 & +9a^2 & \text{Second Power.} \\
+2a & -3a & \\
\hline
+8a^3 & -27a^3 & \text{Third Power.} \\
2a & -3a & \\
\hline
16a^4 & +81a^4 & \text{Fourth Power.}
\end{array}
$$

Where you observe the Powers of $+a$ are all affirmative; and those of $-a$ are alternately negative and affirmative.

186. The Involution of a Compound Quantity is somewhat more operose, but performed after the same Manner, as you observe in the Binomial $a+b$ below.

$a+b$

$a + b =$ Root, or first Power.
$a + b$

$a^2 + ab$
$\quad + ab + bb$

$a^2 + 2ab + b^2 =$ Square, or 2d Power.
$a + b$

$a^3 + 2a^2b + ab^2$
$\quad + a^2b + 2ab^2 + b^3$

$a^3 + 3a^3b + 3ab^2 + b^3 =$ Cube, or 3d Power.
$a + b$

$a^4 + 3a^3b + 3a^2b^2 + ab^3$
$\quad + a^3b + 3a^2b^2 + 3ab^3\,3 + b^4$

$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 =$ 4th Power, &c.

287. The Powers of $a - b$ are raised in the same Manner, Regard being had to the Signs as follows.

$a - b =$ the Root, or first Power.
$a - b$

$a^2 - ab$
$\quad - ab + b^2$

$a^2 - 2ab + b^2 =$ Square, or second Power.
$a - b$

$a^3 - 2a^2b + ab^2$
$\quad - a^2b + 2ab^2 - b^3$

$a^3 - 3a^2b + 3ab^2 - b^3 =$ Cube, or 3d Power, &c.

Here it is easy to see, without proceeding farther, that the several Powers of this Binomial $a - b$ consist of the same Terms as those of the foregoing one $a + b$; the Difference being only in the Signs, where you observe the Sign is *Negative* wherever the Index of b is an *odd Number*, as b, b^3, &c. and affirmative when the said Index is an *even Number* as b^2, b^4, &c.

288. In these Powers of $a+b$ or $a-b$ we are to take Notice of the following Particulars, *viz.* (1) In the first Term of any Power, the Quantity a has the Index of that Power; thus in the 2d Power, it is a^2; in the 4th Power it is a^4, &c. (2) That in the following Terms the Exponents of a decrease gradually by Unity; as in the 4th Power, they are $a^3\ a^2\ a^1$. (3) In the last Term, the Quantity a is not found at all. (4) That the same Things are true of the Quantity b, and Exponents, in a contrary Order; it is not found in the first Term; the Exponents increase by Unity from the 2d to the last Term; and that in the last Term the Exponent is the Index of the Power. (5) That therefore the Sum of the Exponents of both the Quantities in any Term is the same, and always equal to the Exponent or Index of the Power; thus in the 4th Power you find,

$$\text{The Exponents of} \begin{cases} a & 4 \quad 3 \quad 2 \quad 1 \quad 0 \\ b & 0 \quad 1 \quad 2 \quad 3 \quad 4 \end{cases}$$

Whence their Sums are $4+0=3+1=2+2=1+3=0+4=4$, the Exponent of the Power.

289. The Coefficients of the Terms also arise and proceed in a regular Order; and may be found for any Term, or all the Terms, of any Power of a Binomial by a *general Rule*; which, that it may be more obvious to the Reader, it may be proper to place all the Powers of $a+b$ as far as the 6th in one View as below:

$a+b$; first Power, or Root.

$a^2+2ab+b^2$; 2d Power.

$a^3+3a^2b+3ab^2+b^3$; 3d Power.

$a^4+4a^3b+6a^2b^2+4ab^3+b^4$; 4th Power.

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$; 5th. Power.

$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$; 6th Power.

290. Here it may be observed, 1. That when the Number of Terms is odd, the Coefficient of the middle Term is the greatest of all. 2. That the Coefficients on each Side the middle Term are the same, but in an inverse Order. 3. That in those Powers which consist of an even Number of Terms, the Coefficients of the two middle Terms are the same; and also the Coefficients of the Terms on each Side are the same towards the Extremes. 4. That the Coefficients of the first and last Term being Unity, the

the Coefficient for any other Term may be found by the following

RULE.

Multiply the Coefficient of the preceding Term by the Exponent of a *in that Term; and divide the Product by the Exponent of* b *in the given Term; the Quotient is the Coefficient required for that Term.*

Thus, for Example, the Coefficients of the Terms of the 6th Power are found *per* Rule $\frac{1 \times 6}{1} = 6 =$ the Coefficient of the 2d Term; then $\frac{6 \times 5}{2} = 15$, the Coefficient of the 3d Term: Again $\frac{15 \times 4}{3} = 20$ the Coefficient of the 4th or middle Terms. From all which it is plain, that the *Terms*, the *Coefficients*, and the *Signs* of the Terms of any Power of a Binomial may be known without the tedious Operation by a constant Multiplication.

291. But to represent this Matter more generally, let $a + b$ be raised to the Power m; then (by 288.) the Terms without their Coefficients will be a^m, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, $a^{m-4}b^4$, $a^{m-5}b^5$, &c. continued 'till the Exponent of b becomes equal to m, for then the Series must end, as the Exponent of a will, in that Case, be nothing.

292. The Coefficients of the several Terms will be found by the foregoing Rule (190.) to be 1, m, $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, and so on till you have one Coefficient more than there are Units in m.

So that we shall have $\overline{a+b}^m = a^m + ma^{m-1}b + m \times \frac{m-1}{2} a^{m-2}b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4}b^4$, &c.

293. By this general Theorem, or Series, you will find the Terms of any Power very readily by substituting the Exponent of the given Power every where for m. For Example, let the 4th Power of $a + b$ be required; then

$$\text{For the Coefficients}\left\{\begin{array}{c} m = 4 \\ m \times \dfrac{m-1}{2} = 6 \\ m \times \dfrac{m-1}{2} \times \dfrac{m-2}{3} = 4 \\ m \times \dfrac{m-1}{2} \times \dfrac{m-2}{3} \times \dfrac{m-3}{4} = 1 \end{array}\right\}\begin{array}{l} m-1=3 \\ m-2=2 \\ m-3=1 \\ m-4=0 \end{array}$$

Therefore $\overline{a+b}^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

294. If any *Trinomial* or compound Quantity of three Terms, as $a+b+c$, be involved, the Square or 2d Power will be $a^2 + 2ab + bb + 2ac + 2bc + c^2$; where you will observe the Sum of the three first Terms is the Square of $a+b$; and the three last Terms are $2c \times \overline{a+b} + c^2$. Hence any Trinomial may be considered as compounded of a Binomial $a+b$ and a simple Quantity c; *viz.* $\overline{\overline{a+b}+c}^2 = \overline{a+b}^2 + 2c \times \overline{a+b} + c^2$. So that the Terms of the Square of any Trinomial may be known from those of a Binomial without actual Involution. And in the same Manner it is shewn that the Cube of a Trinomial $\overline{a+b+c}^3 = \overline{a+b}^3 + 3c \times \overline{a+b}^2 + \overline{3c^2 \times a+b} + c^3$.

295. We shall conclude with on Example of a Binomial in Numbers and Species together for the Sake of Illustration; let $a = 10$, and $b = 2$;

$$\begin{array}{lclcl}
\text{Then } a+b & = & 10+2 & = & 12 \\
a+b & = & 10+2 & = & 12 \\
\hline
a^2+ab & = & 100+20 & & 24 \\
+ab+b^2 & = & +20+4 & & 12 \\
\hline
a^2+2ab+b^2 & = & 100+40+4 & = & 144 = \text{Square.} \\
a+b & = & 10+2 & & \\
\hline
a^3+2a^2b+ab^2 & = & 1000+400+40 & & 144 \\
a^2b+2ab^2+b^3 & = & 200+80+8 & & 12 \\
\hline
a^3+3a^2b+3ab^2+b^3 & = & 1000+600+120+8 & = & 1728 = \text{Cube.}
\end{array}$$

CHAP.

CHAP. X.

Of EVOLUTION.

296. EVOLUTION is the Reverſe of Involution, and conſiſts in *reſolving of Powers into their Roots*, which Operation is called *Extraction* or *Evolution*. As *Involution* of Roots was performed by *Multiplication of Exponents*; (284.) ſo *Evolution* is made by *dividing the Exponent of the Power by the Index of the Root required*. (See the Reaſon of this in 135.) Thus the *Square Root* of a^2 is $a^{\frac{2}{2}} = a$; of a^3 it is $a^{\frac{3}{2}}$; of a^4, it is $a^{\frac{4}{2}} = a^2$; and the *Cube Root* of a^2 is $a^{\frac{2}{3}}$; of a^3 it is $a^{\frac{3}{3}} = a$; of a^4 it is $a^{\frac{4}{3}}$; of a^6, it is a^2; and ſo on. Alſo the Square Root of $a^2 b^2$ is ab; of $a^4 b^8 c^2$, it is $a^2 b^4 c$, and the *Cube Root* of $a^6 b^3$ is $a^2 b$; of $x^9 y^6 z^{12}$, it is $x^3 y^2 z^4$; and ſo of others.

297. From what has been ſaid with Reſpect to the *Signs* in Involution (279, 230.) it appears, *that any Power that has an affirmative Sign, may have an affirmative or negative Root, when the Index of that Root is an even Number*. Thus the Square Root of $+ a^2$ may be $+ a$, or $- a$; becauſe $+ a \times + a = a^2$, and alſo $- a \times - a = a^2$.

298. Alſo from thence likewiſe it follows *that no Root, whoſe Index is an even Number can be found for a Power with a negative Sign*. (See 285.) Thus the Square Root cannot be extracted from $- a^2$; and therefore ſuch Roots of *Negative Powers* are termed *impoſſible* or *imaginary*; and yet they are ſometimes to be conſidered, and will come into Uſe, as we ſhall ſee further on.

299. If the Root to be extracted is denoted by an *odd* Number; then the *Sign of the Root will be the ſame as that of the Power*. Thus the Cube Root of $- a^3$ is $- a$; and of $+ a^3$ it is $+ a$.

300. When the Index of the required Root will compleatly divide the Exponent of the Power propoſed; then that Root is only *a lower Power of the ſame Quantity*. Thus the Square Root of a^6 is $a^{\frac{6}{2}} = a^3$. and the Cube Root of $a^{12} = a^4$.

201. But otherwiſe, *the Root required will have a Fraction for its Exponent*; thus the Square Root of a^3 is $a^{\frac{3}{2}}$; and the Cube

Root

Root of a^2 is $a^{\frac{2}{3}}$; alſo the Square Root of a itſelf is $a^{\frac{1}{2}}$; and ſo of others, as we ſhewed more generally before.

302. Any Quantity with a *fractional Exponent*, may be conſidered as *an imperfect Power*, or *Surd*; and the Symbol uſed to denote Extraction of *Surd Roots* is this $\sqrt{}$, within which the Power is placed, as $\sqrt{a^3}$, and without or over it, the Index of the Root, as $\sqrt[2]{a^3}$ denotes the Square Root of a^3; and $\sqrt[3]{a^5}$ is the Cube Root of a^5: So that $\sqrt[2]{a^3} = a^{\frac{3}{2}}$; and $\sqrt[3]{a^5} = a^{\frac{5}{3}}$; and univerſally, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

303. Theſe imperfect Powers or Surds, are *multiplied and divided*, like other Powers, *by their Exponents added and ſubtracted:* Thus $a^{\frac{1}{2}} \times a^{\frac{5}{2}} = a^{\frac{1}{2}+\frac{5}{2}} = a^{\frac{6}{2}} = a^3$; and $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{17}{12}} = \sqrt[12]{a^{17}}$. And $a^{\frac{7}{2}} \div a^{\frac{3}{2}} = a^{\frac{7-3}{2}} = a^{\frac{4}{2}} = a^2$.

304. They are likewiſe *involved* and *evolved* in the ſame Manner as perfect Powers. Thus the Square of $a^{\frac{1}{2}}$ is $a^{\frac{1}{2}\times 2} = a^{\frac{2}{2}} = a$; the Square of $a^{\frac{3}{2}}$ is $a^{\frac{3}{2}\times 2} = a^{\frac{6}{2}} = a^3$. On the Contrary, the Square Root of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3}} \div 2 = a^{\frac{2}{3\times 2}} = a^{\frac{2}{6}} = a^{\frac{1}{3}}$; the Cube Root of $a^{\frac{3}{4}} = a^{\frac{3}{4\times 3}} = a^{\frac{1}{4}}$.

305. As to the Roots of *Compound Powers* or Quantities, they are known from the very Form of thoſe Powers, if well conſidered, as they ſtand in the foregoing Tablet (289.) For if the Powers be of a *Binomial* Root $a + b$, and a *Square*, then it will conſiſt of three Terms, *viz.* the Square of a, the Square of b, and twice the Product of both, as a^2, $2ab$, bb; as to the Signs, if they are all affirmative, the Signs of each Part of the Root will be ſo; if the Product has a negative Sign, then the 2d Term of the Root will be *negative* or the Root will be $a - b$. If there are any Coefficients of the Squares a^2, and b^2, they will be Square Numbers, and their Roots will be the Coefficients reſpectively of the Terms of the Root a and b, and the Coefficient of the Product will be the Product of both. Therefore, unleſs a Quantity has all theſe Characters, it cannot be a Square. After the ſame Manner of Enquiry you may diſcover

if

if the proposed Quantity be the *Cube* of any Binomial $a+b$, or $a-b$. It will be to little Purpose to say any Thing here of such Powers as have Trinomial Roots, and indeed, unless any Quantity has apparently the above mentioned Criterions of a perfect Power, it will be in vain to attempt any particular Extraction of Roots at all. If any thing of this Kind is necessary to be done, it may be performed by the general *Formula* delivered for *involving* Roots (292.) and which will equally serve for evolving or extracting them, and that out of any Quantity, even *Surds*, and *infinite Series*.

306. For since in the Expression $\overline{a+b}^m$, the Index ^{m}, if it be an *Integer*, shews the Power of $a+b$; so, on the Contrary, when it is a *Fraction*, it denotes the Root of $a+b$: for *a Root* is only a fractional Power of any Quantity. Therefore if $m=\frac{1}{2}$ then $\overline{a+b}^m = \overline{a+b}^{\frac{1}{2}} = \sqrt{a+b}$; Consequently to extract the Square Root of $a+b$ is only to raise $a+b$ to the Power whose Index is $\frac{1}{2}$; then since $\overline{a+b}^m = a^m + ma^{m-1}b + m \times \frac{m-1}{2} a^{m-2}b^2 + \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3$, &c. and $m=\frac{1}{2}$ we shall have

$$a^m = a^{\frac{1}{2}}$$

$$ma^{m-1}b = \tfrac{1}{2}a^{-\frac{1}{2}}b = \frac{b}{2a^{\frac{1}{2}}}$$

$$m \times m \frac{-1}{2} a^{m-2}b^2 = \tfrac{1}{2} \times -\tfrac{1}{4} a^{-\frac{3}{2}} b^2 = -\frac{b^2}{8a^{\frac{3}{2}}}$$

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 = \tfrac{1}{2} \times -\tfrac{1}{4} \times \tfrac{1}{2} a^{-\frac{5}{2}} b^3 = +\frac{b^3}{16a^{\frac{5}{2}}}, \text{ \&c.}$$

So that $\overline{a+b}^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} - \frac{b^2}{8a^{\frac{3}{2}}} + \frac{b^3}{16a^{\frac{5}{2}}}$, &c.

307. After the same Manner you will find, that $\overline{aa+xx}|^{\frac{1}{2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$, &c. And so if you would extract the Cube Root of a^3+x^3, you will find $\overline{a^3+x^3}|^{\frac{1}{3}} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}$, &c. And thus you may proceed for the proposed Root of any other Binomial Quantity whatsoever.

CHAP.

CHAP. XI.

The RATIONALE *of Extracting the* SQUARE *and* CUBE ROOT.

308. WHEN a Number is given, whose *Square Root* is to be extracted, the first Thing to be done is to *point it*, that is, to place a Point over the first Figure and every other Figure afterwards; which Points resolve the Number into Periods of two Figures each, unless the Number of Figures be odd, for then the last Period can have but one Figure; the Reason of which is, that the Square of each of the nine Digits will produce but two Places of Figures; and 10 is the first Number whose Square will produce *three Places* of Figures; and 100 is the first that will produce *five Places* when squared; and so on. Therefore the Points denote the Number of Figures in the Root. Thus in $\dot{6}4$ there is but one Figure in the Root; in $\dot{1}4\dot{4}$ there are *two*; in $\dot{9}9\dot{8}5\dot{6}$ there are *three*; in $\dot{1}0\dot{0}2\dot{0}0\dot{1}$, there are four Figures in the Root; and so for any other Number. (123. 124.)

309. In extracting the *Cube Root*, the given Number is resolved into Periods of three Figures by the Points placed over the *first*, *fourth*, &c. because the *Cube* of the greatest Digit produces but *three* Places of Figures, or 10 is the first Number whose Cube makes *four* Places; and 100 cubed makes *seven* Places, and so on; therefore, over the *first*, *fourth*, *seventh*, *tenth*, &c. Figure we place a Point, to see how many Figures the Root will consist of.

310. Now since there are always so many Places of Figures in the Root as there are Points, or Periods, in the given Number, the Figures of each Place may be represented by Letters, a, b, c, &c. Thus, if there be but one Period, as 64, there will be but one Figure, *viz.* $8 = a$, and so $64 = a^2$, only. But in a Number of two Periods as $\dot{1}4\dot{4}$, there will be two Figures in the Root, *viz.* $12 = 10 + 2 = a + b$. A Number of three Periods, as $\dot{9}9\dot{8}5\dot{6}$ will have a Root of three Places of Figures, *viz.* $316 = 300 + 10 + 6 = a + b + c$, that is $a = 100$, $b = 10$, and $c = 6$. And thus you may proceed for any larger Number of Places.

311.

311. Again, since $\dot{1}4\dot{4} = \overline{a+b}^2 = a^2 + 2ab + b^2 = a^2 + \overline{2a+b} \times b = 100 + 44$; therefore $a^2 = 100$, and $\overline{2a+b} \times b = 44$; consequently $a = 10$, and $2a = 20$; and since $\overline{20+b} \times b = 44$, it plainly shews that $20)44(2 = b$, because $\overline{20+2} \times 2 = 44$: Therefore the Work, in Symbols and in Numbers, will stand as below.

$$
\begin{array}{rl}
 & \dot{1}4\dot{4} \;(\; 10 \\
 & 100 \;\;\; +2 \\
\hline
20+2 \,\big) & 44 \qquad 12 = \text{Root.} \\
\times 2 \,\big) & 44 \\
\hline
 & \cdot\,\cdot
\end{array}
\qquad
\begin{array}{rl}
 & a^2 + 2ab + bb \;(a \\
 & a^2 \\
\hline
2a+b \,\big) & 2ab + bb \\
\times b \,\big) & 2ab + bb \\
\hline
 & \cdot \quad \cdot
\end{array}
$$

312. When the proposed Number has three Periods, as $\dot{9}98\dot{5}6$, then there will be three Parts in the Root, $a + b + c$; and, therefore, $\overline{a+b+c}^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = \dot{9}98\dot{5}6$; here then $a^2 = 90000$, and $a = 300$, and $2a = 600$; consequently $9856 = 2ab + b^2 + 2ac + 2bc + c^2$; therefore $600\;)\;9856\;(\;10 = b$; and now $\overline{2a+b} \times b = 610 \times 10 = 6100$; therefore $9856 - 6100 = 3756 = 2ac + 2bc + c^2 = \overline{2a+2b+c} \times c = \overline{620+c} \times c$; therefore $620\;)\;3756\;(\;6 = c$; and so $626 \times 6 = 3756$. See the Operation below.

$$
\begin{array}{rll}
 & \dot{9}9\dot{8}5\dot{6} & (\; 300 = a \\
 & 90000 & (\;\;\; 10 = b \\
\hline
600 + 10 \,\big) & 9856 & 310 = a + b \\
\times 10 \,\big) & 6100 & \;\;\;\; 6 = c \\
\hline
620 + 6 \,\big) & 3756 & 316 = a + b + c \\
\times 6 \,\big) & 3756 & \\
\hline
 & \cdot\,\cdot\,\cdot\,\cdot &
\end{array}
$$

313. In the above Operation, the Cyphers being every where omitted, the Work will be contracted, and appear in the com-

mon Form, ſince then we take down but one Period at each Diviſion; for we muſt always make as many Diviſions as there are Places of Figures in the Root, and we can get no more than one at a Time.

$$
\begin{array}{rl}
 & 99856 \;(\; 316 \\
 & 9 \\
\hline
\left.\begin{array}{r} 61 \\ \times 1 \end{array}\right) & \begin{array}{r} 98 \\ 61 \end{array} \\
\hline
\left.\begin{array}{r} 626 \\ \times 6 \end{array}\right) & \begin{array}{r} 3756 \\ 3756 \end{array} \\
\hline
 & \ldots\ldots
\end{array}
$$

By this analytic Proceſs, the Reaſon of extracting the Square Root of any Number, we preſume, is very evident, and particularly of all the Examples and Procedure in (123) and (124.)

314. And in the ſame Manner we demonſtrate the Rule for extracting the Cube Root. Thus, let the Cube Number 1728 be propoſed; there are two Points or Periods, the Root therefore will have two Places or Parts, *viz.* $a + b$, whoſe Cube is $a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 = 1728 = 1000 + 728$. Here 'tis evident $a^3 = 1000$, and ſo $a = 10$. Alſo $3\,a^2\,b + 3\,a\,b^2 + b^3 = 728$, and $3\,a^2 = 300$; therefore $300\,)\,728\,(\,2 = b$, whence we ſhall have $3\,a^2\,b = 600$, $3\,a\,b^2 = 120$, and $b^3 = 8$; conſequently $3\,a^2\,b + 3\,a\,b^2 + b^3 = 600 + 120 + 8 = 728$, the exact Remainder; therefore the Cube Root is $a + b = 10 + 2 = 12$.

315. Let another Example be the Number $13824 = a^3 + 3\,a^2 + 3\,a\,b^2 + b^3 = 13000 + 824$, and 'tis evident (from the Table of Powers, 122.) that the next Cube Number leſs than 13000 is 8000, the Root of which is $20 = a$; therefore $a^3 = 8000$, which ſubducted from the given Number leaves $5824 = 3\,a^2\,b + 3\,a\,b^2 + b^3$. Now $3\,a^2 = 1200\,)\,5824\,(\,4 = b$; therefore we have

$3\,a\;\;b$

$$3a^2b = 4800$$
$$3ab^2 = 960$$
$$b^3 = 64$$

$$5824 = 3a^3b + 3ab^2 + b^3.$$

And consequently the Cube Root is $a + b = 20 + 4 = 24$. From all which it is evident the *common Rule*, given for extracting the Cube Root, is nothing more than the above *Analytical Process in Words*.

Chap. XII.

The Analytic *Doctrine of* ARITHMETICAL PROPORTION.

316. WE have already shewn that *Arithmetical Proportion* is that by which a Series of Numbers increase or decrease, by an *equal Quantity*, thro' the whole Progression, (see Inst. 96, 97.) and of which we have given sufficient Examples in Numbers. We shall now prosecute that Doctrine farther, and in a more general Way by *Analysis*. Thus, suppose the first Term of the Series be (a), and the *Ratio*, or Quantity, by which each Term increases or decreases be (b), then will the second Term be $a + b$, the third $a + 2b$, and the whole Series will appear in this Form, *viz.*

If $\begin{cases} \text{Increasing, } a.\ a+b.\ a+2b.\ a+3b.\ a+4b.\ a+5b, \&c. \\ \text{Decreasing } a.\ a-b.\ a-2b.\ a-3b.\ a-4b.\ a-5b, \&c. \end{cases}$

317. Suppose the last Term of the Series be called x, then in the Series above we have the last Term $a + 5b = x$; then the first Term will be $a = x - 5b$; consequently, if to this Equation you constantly add b on both Sides, you will get the two following Series, of the same Value in each Term, *viz.*

$$a.\quad a+b.\quad a+2b.\quad a+3b.\quad a+4b.\quad a+5b.$$
$$x-5b.\quad x-4b.\quad x-3b.\quad x-2b.\quad x-b.\quad x.$$

318. If now you take the three laſt Terms of the ſecond Series, and place them in an inverted Order under the three firſt Terms of the firſt Series, and then add both together, you will have the Sum of the whole Series, thus

$$\begin{array}{lll} a. & a+b. & a+2b \\ x. & x-b. & x-2b \\ \hline \end{array}$$

$$\text{Sum } a+x.\ a+x.\ \ a+x = 3a+3x = \overline{a+x} \times 3.$$

And ſince this will be the Caſe, let the Number of Terms be what it will, 'tis plain the *Sum total of any Series in Arithmetical Progreſſion is equal to the Sum of the firſt and laſt Terms, multiplied by half the Number of Terms.* Hence, if $n =$ the *Number* of Terms, and $s =$ the *Sum* of the whole Series, we have $s = \frac{n}{2} \times \overline{a+x}$, whence $a + x = \frac{2s}{n}$ and $x = \frac{2s}{n} - a$.

319. Again, ſince b is not in the firſt Term, the Coeffici[illegible] of b will be always $n - 1$; therefore the laſt Term will be $a + \overline{n-1} \times b = x = \frac{2s}{n} - a$, (by 999.) whence we get $s = \frac{2an + n^2b - nb}{2} = \overline{a + \frac{nb-b}{2}} \times n$. Thus, for Inſtance, ſuppoſe the Series were $1 + 2 + 3 + 4 + 5 + 6$, &c. continued to 100; then $a = 1$, $b = 1$, and $n = 100$, whence the Sum of ſuch a Series will be $s = 1 + \frac{100 - 1}{2} \times 100 = 5050$.

320. Suppoſe the firſt Term of the Series be nothing, or a Cypher (0), then in the Equation above $x = \frac{2s}{n} - a$, ſince $a = 0$, we have $s = \frac{nx}{2}$ that is, *the Sum of ſuch a Series is equal to the greateſt Term multiplied by half the Number of Terms.* For Example, the Sum of the Series $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = \frac{10 \times 9}{2} = 45$. This is otherwiſe evident by adding the ſame Series to itſelf, with the Terms in an inverted Order, thus

$0 + 1$

$$\begin{array}{l} 0+1+2+3+4+5+6+7+8+9 \\ 9+8+7+6+5+4+3+2+1+0 \\ \hline \end{array}$$

Sum $9+9+9+9+9+9+9+9+9+9=10\times 9=90.$

The Half of which, therefore, is the Sum of the *single Series*, *viz.* 45.

N. B. The Uſe of this Theorem will be found hereafter to be very conſiderable in many Parts of the mathematical and philoſophical Sciences.

CHAP. XIII.

Of GEOMETRICAL PROPORTION, *or* PROGRESSION.

321. WHEN a Series of Quantities encreaſe by a conſtant Multiplicator, or decreaſe by a common Diviſor, they are ſaid to be in *Geometrical Proportion*, as we have formerly obſerved in *Numbers* (ſee 98.) but ſhall here more generally treat of this Doctrine. The ſaid *Multiplier* or *Diviſor* is called the *Ratio* of the Series, which let be denoted by (r), then will the two Series be as follow

Increaſing; $a.\ ar.\ ar^2.\ ar^3.\ ar^4.\ ar^5.$ &c.

Decreaſing; $a.\ \frac{a}{r}.\ \frac{a}{r^2}.\ \frac{a}{r^3}.\ \frac{a}{r^4}.\ \frac{a}{r^5}.$ &c.

which latter Series is alſo thus expreſſed;

$$a \quad ar^{-1} \quad ar^{-2} \quad ar^{-3} \quad ar^{-4} \quad ar^{-5} \quad \&c.$$

322. Now the Property of ſuch a Series is, *that the Product of the firſt and laſt Terms is equal to the Product of any two others equally diſtant from them; or to the Square of the middle Term, if the Number of Terms be odd.* Thus if we take ſix Terms, we have $a \times ar^5 = ar \times ar^4 = ar^2 \times ar^3 = a^2 r^5$. Or, if we take but five Terms, we ſhall have $a \times ar^4$

$= ar$

$= ar \times ar^3 = ar^2 \times ar^2 = a^2 r^4$, and ſo on. Whence all the other Properties of ſuch a Series, mentioned in Chap. IX. of Arithmetic, are eaſily deduced.

323. Suppoſe the laſt Term of the Series be called y, as $ar^5 = y$, then the firſt Term will be $a = \frac{y}{r^5} = yr^{-5}$ (in the increaſing Series;) the ſecond Term y^{r-4}, the third will be y^{r-3}, and ſo on; ſo that the Series will be doubly expreſſed as below, *viz.*

$$\begin{array}{cccccc} a & ar & ar^2 & ar^3 & ar^4 & ar^5, \text{ \&c.} \\ yr^{-5} & yr^{-4} & yr^{-3} & yr^{-2} & yr^{-1} & y. \end{array}$$

324. Now 'tis evident the Product of the firſt Term in the firſt Series, and laſt Term in the other, will be equal to the Product of any other two Terms, taken at an equal Interval from theſe; that is, $a \times y = ar \times y^{r-1} = ar^2 \times yr^{-2} = ar^3 \times yr^{-3}$, &c. (by 138.)

325. In either Way of expreſſing ſuch a Series, we obſerve, *the Sum of all the Terms, except the firſt, is equal to the Sum of all, except the laſt, multiplied by the common Ratio* (r); for $ar + ar^2 + ar^3 + ar^4 + ar^5$, &c. $= \overline{a + ar + ar^2 + ar^3 + ar^4} \times r$; and therefore let $s =$ the Sum of the Series, and we have $s - a = \overline{s - y} \times r$, (which is the above Theorem in Symbols) and conſequently $s - a = sr - yr$. Whence $sr - s = yr - a$, and ſo $s = \frac{yr - a}{r - 1}$.

326. Since the Exponent of r begins from the ſecond Term, it will always be leſs by 1 than the Number of Terms n; and therefore in the laſt Term it will be $n - 1$; ſo that $y = ar^{n-1} = \frac{ar^n}{r} \therefore yr = ar^n$, and ſo $s = \left(\frac{yr - a}{r - 1} =\right) \frac{ar^n - a}{r - 1}$ Wherefore a, r, and n being given, the Sum of the Series s is eaſily found.

327. In any encreaſing Series, if the firſt Terms be a. ar. ar^2. ar^3, &c. and (y) the laſt Term, then will the four laſt Terms of the Series be yr^{-3} yr^{-2} yr^{-1} y; and thus both

Ends

Ends of the Series may be expressed, without the intermediate Terms. Thus

$$a.\ ar.\ ar^2.\ ar^3,\ \&c.\quad yr^{-3}.\ yr^{-2}.\ yr^{-1}.\ y.$$

And in a decreasing Series they will stand thus,

$$a.\ ar^{-1}.\ ar^{-2}.\ ar^{-3},\ \&c.\quad yr^3.\ yr^2.\ yr.\ y.$$

In this Case we have $s - a = \overline{s - y} \div r$, or $rs - ra = sy$, and so $s = \frac{ra - y}{r - 1} = \frac{yr^n - y}{r - 1}$ (see 325.)

328. If the Number of Terms in such a Series be supposed infinite, we shall have $y = o$; for since $n = \infty$,* $rn - 1$ will be infinite also; and *a finite Quality divided by an infinite one is nothing*, therefore $y = \frac{a}{r^{n-1}} = 0$; and consequently the Sum of such a decreasing Series will be $s = \frac{ar}{r-1}$ which is a finite Sum, though the Number of Terms be infinite. Thus $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \&c. = \frac{1 \times 2}{2 - 1} = 2$; and $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \&c. = \frac{1 \times 3}{3 - 1} = \frac{3}{2} = 1\frac{1}{2}$.

These Theorems also we shall hereafter find of very great Use in Philosophy.

* We shall use the Symbol ∞ for *Infinity*, when there is Occasion.

CHAP. XIV.

Of the Nature, Genesis, and Roots of QUADRATIC EQUATIONS.

329. A *Quadratic Equation* properly signifies no more than the *Square of a simple Equation*. Thus the Square of the simple Equation $x - b = 0$, $(\overline{x - b} \times \overline{x - b}) = x^2 - 2bx + b^2 = 0$, which is a *Quadratic Equation*; whence all

all Equations where the highest Power of the unknown Quantity is a *Square*, or the second Power, are call'd *Quadratics*, however they may differ from this in other Parts. For we are here to observe, that this is the most simple Form of a Quadratic, whose Root consists of two Parts, or Quantities, $x - b = 0$; this Root once obtained, gives the Value of x, *viz.* $x = b$.

330. All other Forms of Quadratics are Rectangles made of two different Roots, or Values of x, as $\overline{x - a} = 0 \times \overline{x - b} = 0$ makes $x^2 - \overline{a + b} \times x + ab = 0$; where it is obvious, *the Coefficient of the second Term is the Sum of the two Roots, viz.* a + b; *and the third Term is the Rectangle of both, viz.* a b.

331. But now to determine the Signs by which the Terms will be in all Cases connected, we must consider the Quality of the Roots, *viz.* whether they are *Affirmative* or *Negative*. In the Equations above, the Roots are both affirmative; for since $x - a = 0$, and $x - b = 0$, it will be $x = a$, and $x = b$. Let the Sum of the Roots be $a + b = s$, and the Rectangle $ab = r$, then when both the Roots are affirmative, the Form of the Quadratic Equation will be $x^2 - sx + r = 0$.

332. If both the Roots are negative, *viz.* $x = -a$, and $x = -b$, then $x + a = 0$, and $x + b = 0$, and the Rectangle of both Equations gives $x^2 + ax + bx + ab = 0$; that is, the Form in all such Cases is $x^2 + sx + r = 0$.

333. If one Root be affirmative, as $x = a$, and the other negative, $x = -b$, then the Equations $x - a = 0$, and $x + b = 0$, multiplied together, produce $x^2 - ax + bx - ab = 0$; now as the Root a or b is the greater Quantity, the Sign of the second Term will be $-$ or $+$, and being ambiguous, it is customary to express both the Signs, thus $x^2 \mp sx - r = 0$; in this Form the last Term is always *negative*; and s = the Difference of the Roots.

334. Hence by observing the Signs of the Equation, you will readily discover when the Roots are affirmative or negative; for if the *Signs change alternately* from + to —, and from — to +, as in the first Form, (331.) then both the Roots are *affirmative*; but if the *Signs are all affirmative*, the Roots are both negative, as in the second Form. (332.) And if there

there is but one Change of the Signs as in the 3d Form, (where there are either +, +, —, or +, —, —,) there will be one affirmative Root, and the other negative of Course.

335. There is another Distinction of the Roots of Equations, *viz.* into *real* and *imaginary*, or *possible* or *impossible*. Thus in the Equation $x^2 - a^2 = 0$, both the Roots are *real*; for since $x^2 = a^2$, it will be either $x = +a$, or $x = -a$, as appears by (297.) But the Equation $x^2 + a^2 = 0$, has no possible Root at all; for here $x^2 = -a^2$, and $x = \sqrt{-a^2}$, which is impossible, as appears by (298.) The Roots, therefore, of a Quadratic Equation are both *possible* or *real*, or both *impossible* or *imaginary* together.

336. When a Question is properly stated, in algebraic Terms, and produces a Quadratic Equation, it will always be of one of the three Forms above-mentioned, *viz.*

$$\text{Either}\left\{\begin{array}{l}1.\ x^2 - sx + r = 0\\ 2.\ x^2 + sx + r = 0\\ 3.\ x^2 \mp sx - r = 0\end{array}\right\}\text{Which are all rectangular Quadratics. (330.)}$$

337. Now if in each of these we transpose the known Quantity r, they will appear in the Form of an imperfect Square.

$$\text{Thus}\left\{\begin{array}{l}1.\ x^2 - sx = -r\\ 2.\ x^2 + sx = -r\\ 3.\ x^2 \mp sx = r\end{array}\right\}\text{The Form of a perfect Quadratic being } x^2 + 2bx + bb = 0.$$

338. In order then to resolve a Quadratic Equation, the 1st Thing to be done is to *complete the Square of it*, which is easily done, if we only observe, that the third Term wanting to make it complete, *is the Square of half the Coefficient of the second Term*. Thus in the complete Square $x^2 - 2bx + bb$, the third Term bb is the Square of half the Coefficient $2b$ of the second Term. Therefore since the Coefficient of the second Term in all the foregoing Forms is s, the Half of which is $\frac{s}{2}$ and the Square of that Half is $\frac{ss}{4}$; which Square, therefore, makes the third Term to be added to each Side of the foregoing Equations, to make each a complete Square on that Side where the

unknown Quantity x is; and when this Addition is made, those Equations will stand as below, *viz.*

$$\left.\begin{array}{l} 1.\ x^2 - sx + \frac{1}{4}s^2 = \frac{s^2}{4} - r \\ 2.\ x^2 + sx + \frac{1}{4}s^2 = \frac{s^2}{4} - r \\ 3.\ x^2 \mp sx + \frac{1}{4}s^2 = \frac{s^2}{4} + r \end{array}\right\} \text{In the Form of perfect Squares, whose Root is } x \pm \tfrac{1}{2}s.$$

339. If now you extract the Root from these Equations, we shall have

$$\left.\begin{array}{l} 1.\ x - \frac{s}{2} = \sqrt{\frac{s^2}{4} - r} \\ 2.\ x + \frac{s}{2} = \sqrt{\frac{s^2}{4} - r} \\ 3.\ x \mp \frac{s}{2} = \sqrt{\frac{s^2}{4} + r} \end{array}\right\} \text{And by transposing } \frac{s}{2} \left\{\begin{array}{l} 1.\ x = \sqrt{\frac{s^2}{4} - r} + \frac{s}{2} \\ 2.\ x = \sqrt{\frac{s^2}{4} - r} - \frac{s}{2} \\ 3.\ x = \sqrt{\frac{s^2}{4} + r} \pm \frac{s}{2} \end{array}\right.$$

340. Since the Equation in any Form always contains two Roots, it is evident the Value of x must ever be expressed, in *two Parts or Terms*, since their Sum gives one Root or Value of x, and their Difference gives the other, which is the Reason why you see them always connected with the double Sign $\pm$.

341. But to put this Matter into a clearer Light, let us investigate the two Roots, a and b, of a *Rectangular Equation*, separately in Numbers, which we can do by having the Sum given, *viz.* $a + b = s$, and their Product $ab = r$. (228.) Thus suppose $x = a = 5$, and again $x = b = 3$, then $\overline{x - a} = 0 \times \overline{x - b} = 0$, gives $x^2 - ax - bx + ab = 0$, or $x^2 - 8x + 15 = 0$.

Here

$$\begin{array}{rl}
\text{Here} & a+b=8 \\
\text{And} & ab=15 \\
\hline
\text{Therefore} & b=8-a \\
\text{And} & b=\frac{15}{a}=8-a \\
\text{And so} & 15=8a-a^2 \\
\text{Transposing all the Terms} & a^2-8a=-15 \\
\text{And compleating the Square} & a^2-8a+16=16-15=1 \\
\text{Extract the Root} & a-4=\sqrt{1} \\
\text{And so} & a=+\sqrt{1}+4=5 \\
\text{And of Course } b\,(=8-a) & =-\sqrt{1}+4=3
\end{array}$$

That is in Words, *The Sum of the Radical Quantity, and Half the Coefficient of the middle Term, is the greater Root* a; *and Half the Coefficient less the Radical Quantity, is the lesser Root* b.

342. Before we leave this Subject, it will be proper to observe, there is yet a more simple Form by which the Root may be expressed, than those in (339.) For since $\sqrt{\frac{s^2}{4}-r}=\sqrt{\frac{s^2-4r}{4}}=\sqrt{\frac{1}{4}\times s^2-4r}=\sqrt{\frac{1}{4}}\times\sqrt{s^2-4r}=\frac{1}{2}\times\sqrt{s^2-4r}=\frac{\sqrt{s^2-4r}}{2}$; therefore $\sqrt{\frac{s^2}{4}-r}+\frac{s}{2}=\frac{\sqrt{s^2-4r}+s}{2}$. Hence the Root of a Quadratic Equation will be either $\frac{\sqrt{s^2-4r}\pm s}{2}$ in the two first Forms, or $\frac{\sqrt{s^2+4r}\pm s}{2}$ in the third. Or more particularly, for every Form of a Quadratic Equation, the Root will be as in the following Table.

Equations.	Roots.
1. $x^2-sx+r=0$	$\frac{\sqrt{s^2-4r}+s}{2}$

2. x

EQUATIONS.	ROOTS.
2. $x^2 + sx + r = 0$	$\frac{\sqrt{s^2 - 4r} - s}{2}$
3. $x^2 - sx - r = 0$	$\frac{\sqrt{s^2 + 4r} + s}{2}$
4. $x^2 + sx - r = 0$	$\frac{\sqrt{s^2 + 4r} - s}{2}$

CHAP. XV.

A Collection of QUESTIONS *producing* QUADRATIC EQUATIONS.

343. IN this Place we mean the Solution of ſuch Queſtions or Problems, as involve not the pure *Square* of the unknown Quantity x, as xx; but the Sum or Difference of this Square, and the Rectangle of the Quantity x, and ſome known Quantity s, as $xx \pm sx$; and theſe are called *Adfected Quadratic Equations:* All of which are to be reduced to one or other of the *four Terms* in the laſt Article, and then the ſought Quantity x is known, and the Queſtion ſolved, as will be illuſtrated by the following Examples.

QUESTION I.

344. What two Numbers are thoſe whoſe Sum is 20, and their Product 36?

SOLUTION.

Let the two Numbers be x, y.

Then their Sum is $x + y = 20$.

Their Product $xy = 36$.

Then we have $\begin{cases} y = 20 - x \\ y = \frac{36}{x} \end{cases}$

And conſequently $\frac{36}{x} = 20 - x$

Which gives $xx - 20x = 36$

There-

Therefore $x^2 - 20x - 36 = 0$, which is a Quadratic of the first Form, where $s = 20$, $r = 36$, and $x = \frac{\sqrt{s^2 - 4r} + s}{2}$ $= \frac{\sqrt{400 - 144} + 20}{2} = 18$; and so $y = 2$: Thus the Sum $18 + 2 = 0$, and the Product $18 \times 2 = 36$.

QUESTION II.

345. What two Numbers are those whose Difference is 24, and the Quotient of the greater divided by the Square of the least is 3?

SOLUTION.

Let the two Numbers be x and y

Then (per Question) $x - y = 24$

And $\frac{x}{yy} = 3$

Which gives this Equation $x = 3y^2$, and $y = \sqrt{\frac{x}{3}}$

Therefore we have $x - 24 = y = \sqrt{\frac{x}{3}}$

Whence, by squaring, $x^2 - 48x + 576 = \frac{x}{3}$

Therefore, $3x^2 - 164x + 1728 = x$

And transposing, $3x^2 - 145x + 1728 = 0$

And dividing by 3, $x^2 - 48.\dot{3}x + 576 = 0$

Here then, $s = 48.\dot{3}$; $r = 576$; and (per Form first) $x = \frac{\sqrt{s^2 - 4r} + s}{2} = 27$, and therefore $y = 3$.

QUESTION III.

346. There are two Numbers, the Sum of their Squares is 2368, and the greater of them is in Proportion to the less as 6 to 1, What are the Numbers?

SOLUTION.

Let $a =$ the greater Number, $e =$ the lesser, and $z = 2368$.

Then

Then	1	$aa + ee = z$ } by the Queſtion
And	2	$a : e :: 6 : 1$ } by the Queſtion
2 ∴	3	$1a = 6e$
3 ⊚ 2	4	$aa = 36ee$
1 — 4	5	$ee = z - 36ee$
5 + 36 ee	6	$37ee = z$
6 ÷ $\overline{37}$	7	$ee = \frac{z}{37} = 64$
7 ɯ 2	8	$e = \sqrt{\frac{2}{37}} = 8$
8 × $\overline{6}$	9	$6e = 6\sqrt{\frac{z}{37}} = 48$
3 and 9	10	$a = 48$

Proof { If $a = 48$
And $e = 8$
$aa = 2304$
$ee = 64$
$aa + ee = 2368$
and $48 : 8 :: 6 : 1$

Question IV.

247. There are three Numbers in continued Proportion, the Sum of the Extreams is 156, and the Mean is 72; What are the two Extreams?

Solution.

That is, ſuppoſe a, m, e, in continued Proportion, and $m = 72$.

Then	1	$a + e = 5156 = s$ } by the Queſtion
	2	$a : m :: m : e$ } Quære a, e, &c.
2 ∴	3	$ae = mm$
1 ⊚ 2	4	$aa + 2ae + ee = ss$
3 × 4	5	$4ae = 4mm$
4 — 5	6	$aa - 2ae + ee = ss - 4mm$
6 ɯ 2	7	$a - e = \sqrt{ss - 4mm}$
1 + 7	8	$2a = s + \sqrt{ss - 4mm}$
8 ÷ 2	9	$a = \frac{s + \sqrt{ss - 4mm}}{2} = 108$ } Or { $a = 48$
1 — 9	10	$e = \frac{s - \sqrt{ss - 4mm}}{2} = 48$ } Or { $e = 108$

QUES-

QUESTION V.

348. There are three Numbers in continued Proportion, their Sum is 74, and the Sum of their Squares is 1924: What are those Numbers?

SOLUTION.

That is, a, e, y, are in continued Proportion.

Then	1	$a+e+y=s=74$	Quære a, e, y.
	2	$aa+ee+yy=z=1924$	
	3	$a:e::e:y$	
$5 \because$	4	$ay=ee$	
$1-e$	5	$a+y=s-e$	
$2-ee$	6	$aa+yy=z-ee$	
$4\times\overline{2}$	7	$2ay=2ee$	
$6+7$	8	$aa+2ay+yy=z+ee$	
$5\,\odot^{2}$	9	$aa+2ay+yy=ss-2se+ee$	
8 and 9	10	$z+ee=ss-2se+ee$	
$10\,\underline{\pm}$	11	$2se=ss-z$	
$11\div 2s$	12	$e=\dfrac{ss-z}{2s}=24$	
5	13	$a+y=s-e=50$	
$13\,\odot^{2}$	14	$aa+2ay+yy=2500$	
$4\times\underline{4}$	15	$4ay=4ee=2304$	
$14-15$	16	$aa-2ay+yy=196$	
$16\,\text{lw}^{2}$	17	$a-y=\sqrt{196}=14$	
$13+17$	18	$2a=50+14=64$	
$18\div\overline{2}$	19	$a=32$	Or $\begin{cases} a=18 \\ y=32 \end{cases}$
$13-19$	20	$y=50-32=18$	

QUESTION VI.

349. There are three Numbers in Arithmetical Progression, the first being added to twice the 2d, and three Times the 3d, their Sum will be 62; and the Sum of all their Squares is 275: What are those Numbers?

SOLUTION.

Suppose	1	a, e, y, in Arithmetical Progression	
And	2	$a+2e+3y=62$	by the Question
	3	$aa+ee+yy=275$	
Then	4	$a+y=2e$ (per Article 136.)	

2 — 4

$2 - 4$	5	$2e + 2y = 62 - 2e$
$5 \div 2$	6	$e + y = 31 - e$
$6 - e$	7	$y = 31 - 2e$
$4 - 7$	8	$a = 41 - 31$
$8 \odot_2$	9	$aa = 16ee - 248e + 961$
$7 \odot^2$	10	$yy = 961 - 124e + 4ee$
$9 + 10$	11	$aa + yy = 20ee - 372 + 1922$
$3 - 11$	12	$ee = 372e - 20ee - 1647$
$12 + 20ee$	13	$21ee = 372e - 1647$
$13 - 372e$	14	$21ee = 372e = - 1647$
$14 + \overline{21}$	15	$ee - \frac{124}{7}e = - \frac{549}{7}$
Comp. Square	16	$ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{549}{7} = \frac{1}{4}$
$16\ lw^2$	17	$e - \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$
$17 + \frac{62}{7}$	18	$e = \frac{62}{7} + \frac{1}{7} = 9$, or $8\frac{5}{7}$ the Mean
18×4	19	$4e = 36$, or $\frac{346}{7}$
8 and 19	20	$a = 36 - 31 = 5$, or $34\frac{6}{7} - 31 = 3\frac{6}{7}$
$18 + \overline{2}$	21	$2e = 18$, or $17\frac{3}{7}$
7 and 21	22	$y = 31 - 18 = 13$, or $31 - 17\frac{3}{7} = 13\frac{4}{7}$

Question VII.

350. There are three Numbers in Arithmetical Progression, the Square of the 1st Term being added to the Product of the other two, is 576; the Square of the mean being added to the Product of the two Extreams makes 612, and the Square of the last Term being added to the Product of the first into the second is 792. What are those Numbers?

Solution.

Suppose	1	a, e, y, in Arithmetical Progression, as before.
Then	2	$aa + ye = 576$ by the Question
	3	$ee + ya = 612$ by the Question
	4	$yy + ae = 792$ by the Question
$1 \because$	5	$a + y = 2e$

5×e

$5 \times e$	6	$ae+ye=2ee$
$2+4$	7	$aa+ye+yy+ae=1368$
$7-6$	8	$aa+yy=1368-2ee$
$3-ee$	9	$ya=612-ee$
9×2	10	$2ya=1224-2ee$
$8+10$	11	$aa+2ya+yy=2592-4ee$
$5 \odot^2$	12	$aa+2ya+yy=4ee$
11 and 12	13	$4ee=2592-4ee$
$13+4ee$	14	$8ee=2592$
$14\div 8$	15	$ee=324$
15 ɯ^2	16	$e=\sqrt{324}=18$, the Mean
.8,	17	$aa+yy=1368-2ee=720$
10,	18	$2ya=1224-2ee=576$
$17-18$	19	$aa-2ya+yy=720-576=144$
1 ɯ^2	20	$a-y=\sqrt{144}=12$
5×20	21	$2a=2e+12=48$
$21\div 2$	22	$a=24$
$5-22$	23	$y=20-24=12$

Or, $\begin{cases} a=12 \\ y=24 \end{cases}$

Question VII.

351. It is required to find two such Numbers, that their Sum being subtracted from the Sum of their Squares, may leave 14, and if their Product be added to their Sum it may make 14?

Solution.

Let a and e be put for the Numbers; and let $y=a+e$

Then $\left\{ \begin{array}{l|l} 1 & aa+ee-y=14 \\ 2 & ae+y=14 \end{array} \right\}$ by the Question.

$1+y$	3	$aa+ee=14+y$
$2-y$	4	$ae=14-y$
4×2	5	$2ae=28-2y$
$3+5$	6	$aa+2ae+ee=42-y$
6 ɯ^2	7	$a+e=\sqrt{42-y}$
But	8	$a+e=y$, as above
7 and 8	9	$y=\sqrt{42-y}$
$9\odot^2$	10	$yy=42-y$
$10+y$	11	$yy+y=42$
Sq. compl.	12	$yy+y+\frac{1}{4}=42+\frac{1}{4}=42,25$
12 ɯ^2	13	$y+\frac{1}{2}=\sqrt{42,25}=6,5$
$13-\frac{1}{2}$	14	$y=6,5-\frac{1}{2}=6$

T Con-

Consequ.	15	$a + e = 6$, (for $a + e = y$) as above
3 and 14	16	$aa + ee = 14 + 6 = 20$
5 and 15	17	$2ae = 28 - 12 = 16$
$16 - 17$	18	$aa - 2ae + ee = 4$
8 ɯ^{2}	19	$a - e = \sqrt{4} = 2$
$15 + 19$	20	$2a = 8$
$23 \div 2$	21	$a = 4$
$1 - 21$	22	$e = 6 - 4 = 2$

Question VIII.

352. Three Merchants join Stocks together; the first Man's Stock was less than the second Man's 13*l.* the second and third Man's Stock was 175*l.* In trading they gain 48*l.* more than their whole Stock was; the first Man's proportional Part of the Gain was 78*l.* What was each Man's Stock and Part of the Gain?

Solution.

Let a, e, y, represent each Man's Stock.

Then	1	$a + e + y = s$, the whole Stock.
	2	$s + 48 =$ the whole Gain.
And	3	$a + 13 = e$
	4	$e + y = 175$
$4 + a$	5	$a + e + y = 175 + a$
1 and 5	6	$s = 175 + a$
6 and 2	7	$s + 48 = 223 + a$
But	8	$175 + a : 223 + a :: a : 78$, per Question
$8 \because$	9	$aa + 223a = 78a + 13650$
$9 - 78a$	10	$aa + 145a = 13650$
	11	$aa + 145a + 5256,25 = 18906,25$
11 ɯ^{2}	12	$a + 72,5 = \sqrt{18906,25}$
$12 - 72,5$	13	$a = 137,5 - 72,5 = 65$
3,	14	$e = a + 13 = 78$
$4 - 14$	15	$y = 97$
Then	16	$65 : 78 :: 78 : 93$*l.* 12*s.* e's Gain
Again	17	$65 : 78 :: 97 : 116$*l.* 8*s.* y's Gain
Proof	18	116*l.* 8*s.* + 93*l.* 12*s.* + 78*l.* = 288, the Gain
	19	$65 + 78 + 97 = 240$, the whole Stock
$18 - 19$	20	$288 - 240 = 48$, the Gain more than the Stock.

353. These

353. These Examples will, for the present, be sufficient for *Quadratic Equations*, according to the best Methods of Solution. What relates to adfected *Cubic*, and *higher Equations*, we shall defer 'till after we have prosecuted the Doctrine of *simple and compound Interest*, which now the Reader is prepared for; since if Equations, higher than *Quadratics*, occur in that Affair, they are *simple ones*, and may be solved by *Logarithms*, or *Tables* made for that Purpose.

CHAP. XVI.

Of ANNUITIES, PENSIONS, *&c. in* Arrears, *at* SIMPLE INTEREST.

354. ANNUITIES, or Pensions, *&c.* are said to be in Arrears, when they are payable or due, either yearly, or half-yearly, *&c.* and are unpaid for any Number of Payments. Therefore the Business is to compute what all those Payments will amount unto, allowing any Rate of simple Interest for their Forbearance, for the Time each particular Payment became due: Now, in order to that,

$$\text{Put}\begin{cases} U = \text{the Annuity, Pension, or yearly Rent, \&c.} \\ T = \text{the Time of its Continuance, or being unpaid.} \\ R = \text{the } \textit{Ratio}\text{, or Interest of 1}\textit{l.}\text{ for one Year, as before.} \\ A = \text{the Amount of the Annuity and its Interest.} \end{cases}$$

355. Then say, as $1l. : R :: U : RU$, the Interest due at the End of the second Year, for the Rent forborn one Year; and $2U$ will be the Rent or Annuity due at the same Time. So that for any given Time, the Interests and Rents will stand as below for each successive Year, *viz.*

$$\left.\begin{matrix} RU = \text{the Interest} \\ 2U = \text{the Rent} \end{matrix}\right\} \text{due at the End of the 2d Year.}$$

$$\left.\begin{matrix} 2RU = \text{the Interest} \\ 3U = \text{the Rent} \end{matrix}\right\} \text{due at the End of the 3d Year.}$$

$\left.\begin{array}{r} 3RU = \text{the Interest} \\ 4U = \text{the Rent} \end{array}\right\}$ due at the End of the 4th Year.

$\left.\begin{array}{r} 4RU = \text{the Interest} \\ 5U = \text{the Rent} \end{array}\right\}$ due at the End of the 5th Year.

And ſo on, for any Number of Years. Hence it is evident, that $RU + 2RU + 3RU + 4RU + 5RU = A$, the Sum of all the Rents and their Intereſt, being forborne five Years.

356. Let $T =$ the Time or Number of Years, then it follows that $RU + 2RU + 3RU + 4RU = A - TU$. Here $T = 5$; divide by U; then $R + 2R + 3R + 4R = \frac{A - TU}{U}$. Next to find the Sum of this Progreſſion: Thus, Let $R + 2R + 3R + 4R$, &c. $= Z$; then $1 + 2 + 3 + 4$, &c. $= \frac{Z}{R}$. Here the Sum of the firſt and laſt Terms are $4 + 1 = 5 = T$, and the Number of all the Terms is $4 = T - 1$: Therefore $\frac{T - 1}{2} \times T =$ the Sum of all the Terms; that is, $\frac{TT - T}{2} = \frac{Z}{R}$: Hence $\frac{TTR - TR}{2} = Z$, conſequently $\frac{TTR - TR}{2} = \frac{A - TU}{U}$.

What has been here ſaid may be eaſily deduced from Chap. 12, and Theorem in Article 318.

357. From the laſt Equation we raiſe the following Theorems. Theorem I. $\frac{TTU - TU}{2} \times R + TU = A$.

Theorem II. $\frac{2A}{TTR - TR + 2T} = U$. Theorem III. $\frac{2A - 2TU}{TT - TU} = R$. (Let $\frac{2}{R} - 1 = x$.) Then Theorem IV. $T = \sqrt{\frac{2A}{RU} + \frac{xx}{4}} - \frac{1}{2}x$.

Question I.

358. If 250*l.* yearly Rent, (Penſion, &c.) be forborne or unpaid ſeven Years; what will it amount to in that Time, at 3 *per Cent.* for each Payment, as it becomes due?

Here

Here is given $\left\{\begin{array}{l} U = 250 \\ T = 7 \\ R = 0,03 \end{array}\right\}$ To find A, the Amount.

By Theorem I. $\frac{TTU - TU}{2} \times R + TU = \frac{7 \times 7 \times 250 - 7 \times 250}{2}$ $\times 0,03 + 7 \times 250 = \frac{10500}{2} \times 0,03 + 7 \times 250 = 157,5$ $+ 1750 = 1907,5 = 1907$*l.* 10*s.* = the Amount required.

359. If the Annuity or Pension were to be paid *half-yearly*; then $U = 125$, $T = 14$, the Number of Payments, and $R = \frac{0,03}{2} = 0,015$, to find A; then the Theorem will be as follows:

$$TTU = 14 \times 14 \times 125 = 24500$$
$$TU = 14 \times 125 = 1750$$
$$TTU - TU = 24500 - 1750 = 22750$$
$$\frac{TTU - TU}{2} \times R = \frac{22750}{2} \times 0,015 = 170,625$$

$170,625 + TU = 170,625 + 1750 = 1920,625 = A =$ 1920*l.* 12*s.* 6*d.*

Wherefore the half-yearly Payment is more advantageous than the yearly one by 13*l.* 2*s.* 6*d.* and consequently quarterly Payments are still more so than these.

Question II.

360. What Annuity, Pension, *&c.* being forborn, or unpaid seven Years, will raise a Stock of 1907*l.* 10*s.* at the Rate of 3 *per Cent.* Interest for each Payment as it becomes due?

Here $\left\{\begin{array}{l} A = 1907,5 \\ T = 7 \\ R = 6,03 \end{array}\right\}$ to find U, the Annuity, *&c.*

Then per Theorem II. $\frac{2A}{TTR - TR + 2T} =$ $\frac{2 \times 1907,5}{7 \times 7 \times 0,03 - 7 \times 0,03 + 2 \times 7} = 250$*l.* the Annuity required.

QUES-

Question III.

361. If 250*l.* yearly Rent, being forborne seven Years, will amount to 1907*l.* 10*s.* allowing simple Interest for every Payment as it becomes due, what must the Rate of Interest be *per Cent*, &c.

$$\text{Here}\left\{\begin{array}{l} U = 250l. \\ A = 1907{,}5l. \\ T = 7 \text{ Years} \end{array}\right\}\text{to find R, the Rate of Interest.}$$

Per Theor. III. $\frac{2A-2TU}{TTU-TU}=R=\frac{2\times1907{,}5-2\times7\times250}{7\times7\times250-7\times250}$ $=0{,}03$; then $1:0{,}03::100l.:3l.$ the Interest *per Cent.* required.

Question IV.

362. In what Time will 250*l.* yearly Rent, amount to 1907*l.* 10*s.* at 3*l.* *per Cent.* for Forbearance of Payment as they become due?

$$\text{Here}\left\{\begin{array}{l} U = 250l. \\ A = 1907{,}5 \\ R = 0{,}03 \end{array}\right\}\text{to find T, the Time, per Theor. II.}$$

Here $\frac{2}{R}-1=x=\frac{2}{0{,}03}-1=65{,}\not{6}$; and $\frac{1}{2}x=32{,}8\not{3}$; then $\sqrt{\frac{2A}{RU}+\frac{x^2}{4}}-\frac{1}{2}x=T=\sqrt{\frac{2\times1907{,}5}{0{,}03\times250}-\frac{65{,}\not{6}\times65{,}\not{6}}{4}}$ $-32{,}8\not{3}=7$, the Time, or Number of Years required.

Chap. XVII.

The Present Worth *of* Annuities, Pensions, *&c. computed at* Simple Interest.

363. THE Business of purchasing Annuities, or taking of Leases, *&c.* for any assigned Time, depends upon the true equating of the Principal, or Money, laid out on the Pur-

Purchaſe, with the Annuity or yearly Rent, by allowing (or diſcounting) the ſame Rate of Intereſt to both Parties; which may be eaſily performed, by duly applying the reſpective Theorems in Article 270 and 357, for finding the Amount of any Annuity and its Intereſt, forborne for any given Time; and then what Sum or Principal, put to Intereſt at the ſame Rate, will in the ſame Time amount to the ſame Sum.

364. Now ſince the Amount A in both Caſes is the ſame, the Theorems which give the ſaid Amount, muſt be equal to each other; that is, (Article 357. Theorem I.) $\frac{ttU - tU}{2} \times R + tU = tPR + P = A$. (Article 270.) Which two Theorems compounded together give the following, *viz.*

$$\text{Theorem I. } \frac{ttR - tR + 2t}{2tR + 2} \times U = P.$$

$$\text{Theorem II. } \frac{tR + 1}{ttR - tR + 2t} \times 2P = U$$

$$\text{Theorem III. } \frac{2P - 2tU}{ttU - tU - 2Pt} = R$$

Now put $\frac{2}{R} - \frac{2P}{U} - 1 = x$; then $tt \pm xt = \frac{2P}{RU}$, which gives

$$\text{Theorem IV. } t = \sqrt{\frac{2P}{RU} + \frac{xx}{4}} \pm \frac{x}{2}$$

QUESTION I.

365. What is 75 *l.* yearly Rent, to continue nine Years, worth in Ready Money, at 3 *per Cent.*

$$\text{Here} \left\{ \begin{array}{l} U = 75l. \\ R = 0{,}03 \\ t = 9 \text{ Years} \end{array} \right\} \text{to find the preſent Worth P.}$$

Theor. I. $\frac{ttR - tR + 2t}{2tR + 2} \times U = \frac{9\times9\times0{,}03 - 9\times0{,}03 + 2\times9}{2\times9\times0{,}03 + 2} \times 75 = \frac{2{,}43 - 0{,}27 + 18}{0{,}54 + 2} \times 75 = 611l.\ 4s.\ 4d.\frac{3}{4}$. the preſent Worth required.

QUES-

Question II.

366. What Annuity, to continue 21 Years, will 192*l.* 1*s.* 5½*d.* purchase, at 5 *per Cent?*

$$\text{Here}\left\{\begin{array}{l}P = 192{,}0731\,l.\\ t = 21 \text{ Years}\\ R = 0{,}05\end{array}\right\}\text{to find the Annuity U.}$$

Then, Theor. II. in Numbers, $\dfrac{21 \times 0{,}05 + 1}{21\times 21\times 0{,}05 - 21\times 0{,}05 + 2\times 21} \times 2 \times 192{,}0731 = 12{,}5 = 12\,l.\ 10\,s.$ the Annuity required.

Question III.

367. At what Rate of simple Interest will 192*l.* 1*s.* 5½*d.* purchase an Annuity of 12*l.* 10*s.* to continue 21 Years?

$$\text{Here}\left\{\begin{array}{l}P = 192{,}0731\,l.\\ U = 12{,}5\,l.\\ T = 21 \text{ Years}\end{array}\right\}\text{to find R.}$$

Per Theor. III. $\dfrac{2 \times 192{,}0731 - 2 \times 21 \times 12{,}5}{21\times 21 \times 12{,}5 - 21\times 12{,}5 - 2\times 21 \times 192{,}0731} = 0{,}05$; then $1 : 0{,}05 :: 100\,l. : 5\,l.$ the Interest *per Cent.* allowed.

Question IV.

368. In what Time will 7*l.* *per Ann.* pay a Debt of 120*l.* 8*s.* at 6*l.* *per Cent.* Interest? Or thus; For how long a Time may an Annuity of 7*l.* *per Ann.* be purchased, or enjoyed, for 120*l.* 8*s.* at the aforesaid Rate of Interest?

$$\text{Here}\left\{\begin{array}{l}U = 7\,l.\\ R = 0{,}06\\ P = 120{,}4\,l.\end{array}\right\}\text{to find T, the Time.}$$

Then $\dfrac{2}{R} - \dfrac{2P}{U} + 1 = x = 2{,}0\phi.$ (See Article 364.) And $\sqrt{\dfrac{2P}{RU} + \dfrac{xx}{4}} + \frac{1}{2}x = T = 24{,}\phi = 25$ Years, the Time required.

CHAP.

Chap. XVIII.

The Construction and Use of Tables of Simple Interest.

369. THE great Design of Tables of Interest (both Simple and Compound) is Ease and Expedition in practical Calculations. For, besides that the Rules expressed in Words for answering Questions of Interest are tedious and intricate, and the Reason no Ways to be understood, the Operations themselves are, for the most Part, very laborious; and consequently Tables which expedite and facilitate the Practice are indispensibly necessary.

370. These Tables are made in Decimal Numbers, the first for Days, and the second for Years; which Numbers, being Arithmetical Proportion, make them capable of that Perfection which no other Tables can pretend to: They are so contrived, that the Interest of any principal Sum is easily found for any Number of Days, or Years, at any Rate, from one Pound to ten, with the Halves and Quarters; having followed herein the Rev. Mr. *Brown* in his *Arithmetica Infinita*.

371. The Construction of these Tables is easy from the Theorems themselves, (and indeed the Reason of their Construction can be no otherways so easily conceived.) Thus, by Theorem I. of Simple Interest, *viz.* $tRP + P = A$ is the first and second Table constructed. For since the Amount less the Principal, is equal to the Interest, therefore the Theorem will be tRP = Interest. Now if $P = 1l.$ $t = ,002739$, &c. (the Decimal of a Year for one Day) and R = any Ratio of Interest; suppose 5 *per Cent.* then the Simple Interest of one Pound for one Day, at 5 *per Cent.* is $,002739$, &c. $\times ,05 \times 1 = 00013698$, &c. which being multiplied by the nine Digits, severally constitute that Part of the Table of Interest at 5 *per Cent.* and thus the whole first Table is made. The second Table, for Years, is only the various *Ratios* of Interest multiplied by the said nine Digits; for since $t = 1$ Year, and $P = 1l.$ it will be $tRP = R$, the Interest for the first Year, &c.

372. The third Table ſhews the Rebate or Diſcount to be made for one Pound, at the ſeveral Rates *per Cent.* for Days. The Manner, Truth, and Reaſon of its Conſtruction is derived from Theorem II. of Simple Intereſt, *viz.* $\frac{A}{tR+1}=P$. For ſince the Principal or preſent Worth, ſubducted from the Amount, gives the Rebate or Diſcount of that Amount; therefore the Diſcount of any Amount for any Time, at any Rate, (without Regard of the preſent Value, or principal Money) may be found by this Theorem $\frac{AtR}{tR+1}=D=$ Diſcount. $\left(\text{For } A-\frac{A}{tR+1}=\frac{AtR}{tR+1}.\right)$ Hence if we put $A=1l.$ $t=,002739$, &c. and $R=$ any *Ratio* of Intereſt, ſuppoſe 5 *per Cent.* then by this laſt Theorem we have the Diſcount of one Pound for one Day, at the Rate of 5 *per Cent. per Ann.* for $AtR=1\times,002739$, &c. $\times,05=,00013698$, &c. And $tR+1=1,00013698$, &c. then by Diviſion, 1,00013698, &c.) ,00013698, &c. ($=,00013697$, &c. the Diſcount. If $t=1$ Year, then the annual Diſcount of one Pound at 5 *per Cent.* will be found, by the above Theorem, thus; $AtR=,05$, and $tR+1=1,05$. Therefore by Diviſion, 1,05) ,05 ($=,04761904$, &c. the Diſcount. And thus is the Diſcount of any Sum, at any Rate for any Time above one Year, found at once by the above Theorem; and for any Time under a Year by the Table of Diſcount for Days, of which I have now taught the Conſtruction in a new and more rational Method than any I have yet ſeen.

The Uſe of Table I. *and* II.

373. In order to underſtand how to make thoſe two Tables univerſally uſeful, the Reader is to obſerve, that if a Number conſiſts of only one Digit, with Cyphers affixed, as 10, 50, 700, 9000, 800000, &c. it is called a *pure Number*; but thoſe Numbers which conſiſt of more than one, or wholly of Digits, as 370, 568, 7569, &c. may be called *mixed Numbers.* Now every mixed Number may be reſolved into thoſe pure Numbers, of which they are compoſed; thus the mixed Number 567 may be

be resolved into the pure Numbers 500, 60, and 7; so also 15890 is resolved into 10000, 5000, 800, and 90.

374. Then as to the Use of the Tables, observe these Rules:

I. If the Number of Days, Years, &c. proposed, be a mixed Number, let it be resolved into pure Numbers.

II. With the pure Numbers severally enter the Tables, and take those Decimal Numbers which stand against the first Figure of each pure Number, in the Column marked *Numbers*.

III. Remove the Decimal Point in each such Decimal Number, so many Places to the Right Hand, as there are Cyphers in the respective pure Numbers.

IV. Lastly, Add together all the Decimal Numbers, and find the Value thereof by the Tables for that Purpose.

These Things premised, the Use of the Tables will be obvious from the Examples of the following Problems.

Problem I.

375. To find the Interest of any Sum of Money for a Day, or a Year, at any Rate *per Cent. per Ann.*

Example I.

What is the Interest of 2746*l.* at 5*l.* 15*s.* *per Cent.* for a Day?

		Decimals.
In Table I. under the Rate $5\frac{3}{4}$, you find against the pure Numbers	2000 —	,31506
	700 —	,11027
	40 —	,00630
	6 —	,00094
The Answer is	—	,43257 = 8*s.* $7\frac{3}{4}$*d.*

Example II.

376. What is the Interest of the same Sum, at the same Rate for a Year?

		Decimals.		
In Table II. you find against	2000 —	115,00000	Under $5\frac{3}{4}d.$ *per Cent.*	
	700 —	40,25000		
	40 —	2,30000		
	6 —	0,34500		

The Anſwer in Decimals *l.* 157,895

Which is in Money = 157*l.* 17*s.* $10\frac{3}{4}d.$

Problem II.

377. To find the Intereſt of any Sum of Money for any Number of Days.

Example.

What is the Intereſt of 265*l.* for 149 Days, at the Rate of 3*l.* 15*s.* *per Cent.* &c.?

Multiply the principal Sum	—	265*l.*
By the given Number of Days	—	149

The Product is the mixed Number 39485, with which reſolved, enter the Table as before:

		Decimals.	
Thus in Table I. you find againſt the pure Numbers	30000 —	3,08220	Under $3\frac{3}{4}d.$ *per Cent.*
	9000 —	0,92466	
	400 —	0,04109	
	80 —	0,00822	
	5 —	0,00051	

The Anſwer in Decimals *l.* 4,05668

In Money 4*l.* 1*s.* $1\frac{1}{2}d.$

The Method is the ſame for any greater Number of Days.

Problem III.

378. To find the Intereſt of any Sum forborne any Number of Years, at any of the given Rates *per Cent.*

Example.

What is the Intereſt of 175*l.* 15*s.* forborne 13 Years, at the Rate of 6 *per Cent.* &c.?

Mul-

Multiply the principal Sum	—	175,75
By the Number of Years given	—	13
The Product is the mixed Number		2284,75

Which resolved, as before, will stand thus,

In Table II. you find against	2000,	—	120,000	Under 6 *per Cent.*
	200,	—	12,000	
	80,	—	4,800	
	4,	—	0,240	
	,7	—	0,042	
	,05	—	0,003	

The Answer in Decimals *l.* 137,085

The same in Money 137*l.* 1*s.* 8$\frac{1}{4}$*d.*

N. B. The Reader must observe, in resolving a mixed Number, wherein are Decimals, to remove the Point one Place more to the *left*, than are the Number of Cyphers in the Decimal pure Number, as in the last Example.

The Use of Table III. *Of* Discount.

379. In seeking the Discount for any Sum due at the End of any Number of Days, if the Number of Days be a mixed one, resolve them into pure Numbers as before taught; and even with them in the Table take the Discount of 1*l.* which add together; and then multiply by the principal Sum, the Product will be the Discount thereof.

Problem IV.

To find the Discount of any Sum, for any Number of Days, at any given Rate in the Table.

Example.

What is the Discount of 83*l.* 10*s.* for 235 Days, at 4 *per Cent. per Annum?*

Deci-

			Decimals.	
You find in the Table even with	200	—	,0214478	Under 4 *per Cent.*
	30	—	,0032769	
	5	—	,0005476	
The Sum is		—	,0252723	
Which multiplied by the Sum			83,5	

		l.	s.	d.
The Product is the Answer	2,110237 =	2 —	2 —	$2\frac{1}{4}$

PROBLEM V.

380. To find the Discount of any Sum for a Year.

EXAMPLE.

What is the Discount of 100*l.* for one Year, at 5 *per Cent?*

		l.	s.	d.
In the Table under 5 *per Cent.* and against 365 Days, is	,047619, *&c.*			
Which multiplied by the principal Sum ——	100			
The Product is the Answer	*l.* 4,7619, *&c.* =	4 —	15 —	$2\frac{3}{4}$
Now the Interest of 100*l.* for one Year, at 5 *per Cent.* is — — — —		5 —	0 —	0
The Difference therefore of Discount and Interest is — — — —		0 —	4 —	$9\frac{1}{4}$

Whence it is evident, he who allows Interest for Discount wrongs himself considerably, which yet is very common among Traders; for so much Money ought to be paid as, at Interest, would amount to the Sum due, in the Time proposed.

EXAMPLE II.

381. What is the Discount of 9342*l.* at $4\frac{1}{2}$ *per Cent.* for a Year?

The

The Discount of 1*l.* for 365 Days, at $4\frac{1}{2}$ *per Cent.* in the Table, is } ,043062, &c.
Which multiplied by the principal Sum 9342

The Product is the Answer — *l.* 402,2852, &c.
In Money 402*l.* 5*s.* $8\frac{1}{2}$*d.*

And thus proceed for other annual Discounts.

It must be acknowledged this Table of Discount gives not the precise Truth, and yet differs but little from it; being sufficiently exact for any Use. None but a Table of the Discount for every Day can be perfect; because every Day's Discount differs, being still less as the Number of Days increase.

This Table is perfectly true for all the Days expressed therein, and, as I said, may be used without much Error for any other.

		l.	*s.*	*d.*
Example in Prob. IV.	The true Discount is —	2	1	11
	The Discount by this Table	2	2	$2\frac{1}{4}$
	The Interest for the Time and Rate	2	3	0

TABLES

TABLES OF SIMPLE INTEREST.

TABLE I.

The Interest of one Pound per Diem, *at any Rate* per Cent. *from one to ten Pounds, with Halves and Quarters.*

Numb.	1 *per Cent.*	$1\frac{1}{4}$ *per Cent.*	$1\frac{1}{2}$ *per Cent.*	$1\frac{3}{4}$ *per Cent.*
1	,00002740	,00003425	,00004110	,00004794
2	,00005480	,00006850	,00008220	,00009589
3	,00008220	,00010274	,00012329	,00014383
4	,00010959	,00013699	,00016438	,00019178
5	,00013698	,00017123	,00020548	,00023972
6	,00016438	,00020548	,00024657	,00028767
7	,00019178	,00023973	,00028767	,00033562
8	,00021918	,00027398	,00032877	,00038356
9	,00024657	,00030822	,00036986	,00043151
Month.		,00104166	,00125000	,00145833

Numb.	2 *per Cent.*	$2\frac{1}{4}$ *per Cent.*	$2\frac{1}{2}$ *per Cent.*	$2\frac{3}{4}$ *per Cent.*
1	,00005480	,00006164	,00006849	,00007534
2	,00010959	,00012329	,00013699	,00015068
3	,00016438	,00018493	,00020547	,00022602
4	,00021918	,00024657	,00027397	,00030137
5	,00027397	,00030822	,00034246	,00037671
6	,00032876	,00036986	,00041095	,00045205
7	,00038356	,00043151	,00047945	,00052739
8	,00043835	,00049315	,00054794	,00060274
9	,00049315	,00055479	,00061644	,00067808
Month.	,00166666	,00187500	,00208333	,00229166

Numb.

Numb.	3 per Cent.	$3\frac{1}{4}$ perCent.	$3\frac{1}{2}$ per Cent.	$3\frac{3}{4}$ perCent.
1	,00008220	,00008904	,00009589	,00010274
2	,00016438	,00017808	,00019178	,00020548
3	,00024657	.00026712	,00028767	,00030822
4	,00032877	,00035616	,00038356	,00041096
5	,00041096	,00044520	,00047945	,00051363
6	,00049315	,00053424	,00057534	,00061644
7	,00057534	,00062328	,00067123	,00071917
8	,00065753	,00071232	,00076712	,00082192
9	,00073972	,00080137	,00086301	,00092465
Month.	,00250000	,00270833	,00291666	,00312500

Days.	4 per Cent.	$4\frac{1}{4}$ perCent.	$4\frac{1}{2}$ perCent.	$4\frac{3}{4}$ perCent.
1	,00010959	,00011644	,00012329	,00013014
2	,00021918	,00023288	,00024657	,00026027
3	,00032877	,00034931	,00036986	,00039041
4	,00043836	,00046575	,00049315	,00052055
5	,00054794	,00058219	,00061643	,00065068
6	,00065753	,00069863	,00073973	,00078082
7	,00076712	,00081507	,00086301	,00091096
8	,00087671	,00093151	,00098630	,00104109
9	,00098630	,00104794	,00110959	,00117123
Month.	,00333333	,00354166	,00375000	,00395833

Days.	5 per Cent	$5\frac{1}{4}$ perCent.	$5\frac{1}{2}$ perCent.	$5\frac{3}{4}$ perCent.
1	,00013698	,00014383	,00015068	,00015753
2	,00027397	,00028767	,00030137	,00031507
3	,00041096	,00043151	,00045205	,00047260
4	,00054794	,00057534	,00060274	,00063014
5	,00068493	,00071918	,00075342	,00078767
6	,00082192	,00086301	,00090411	,00094520
7	,00095890	,00100685	,00105479	,00110274
8	,00109589	,00115068	,00120548	,00216027
9	,00123288	,00129452	,00135616	.00141781
Month.	,00416666	,00437500	,00458333	'00479166

Days.	6 *per Cent.*	6¼ *perCent.*	6½ *perCent.*	6¾ *perCent.*
1	.00016438	.00017123	.00017808	.00018493
2	.00032876	.00034246	.00035616	.00036986
3	.00049315	.00051370	.00053424	.00055479
4	.00065753	.00068493	.00071232	.00073972
5	.00082192	.00085616	.00089041	.00092466
6	.00098630	.00102740	.00106849	.00110959
7	.00115068	.00119863	.00124657	.00129452
8	.00131507	.00136986	.00142465	.00147945
9	.00147945	.00154109	.00160274	.00166438
Month.	.00500000	.00520833	.00541666	.00562500

Days.	7 *per Cent.*	7¼ *per Cent.*	7½ *perCent.*	7¾ *perCent.*
1	.00019178	.00019863	.00020548	.00021233
2	.00038356	.00039726	.00041096	.00042466
3	.00057534	.00059589	.00061644	.00063699
4	.00076712	.00079452	.00082192	.00084932
5	.00095890	.00099315	.00102739	.00106164
6	.00115068	.00119178	.00123288	.00127397
7	.00134246	.00139041	.00143836	.00148630
8	.00153425	.00158904	.00164384	.00169863
9	.00172603	.00178767	.00184932	.00191096
Month.	.00583333	.00604166	.00625000	.00645833

Days.	8 *per Cent.*	8¼ *perCent.*	8½ *perCent.*	8¾ *perCent.*
1	.00021918	.00022603	.00023287	.00023973
2	.00043835	.00045205	.00046575	.00047945
3	.00065753	.00067808	.00069863	.00071918
4	.00087671	.00090411	.00093150	.00095890
5	.00109589	.00113014	.00116438	.00119863
6	.00131507	.00135616	.00139726	.00143835
7	.00153425	.00158219	.00163013	.00167808
8	.00175342	.00180822	.00186301	.00191781
9	.00197260	.00203424	.00209589	.00215753
Month.	.00666666	.00687500	.00708333	.00729166

Days.	9 *per Cent.*	$9\frac{1}{4}$ *per Cent.*	$9\frac{1}{2}$ *per Cent.*	$9\frac{3}{4}$ *per Cent.*
1	.00024657	.00025342	.00026028	.00026712
2	.00049315	.00050684	.00052055	.00052424
3	.00073972	.00076027	.00078082	.00080137
4	.00098630	.00101370	.00104109	.00106849
5	.00123287	.00126712	.00130137	.00133561
6	.00147945	.00152055	.00156164	.00160274
7	.00172602	.00177397	.00182192	.00186986
8	.00197260	.00202739	.00208219	.00213699
9	.00221918	.00228082	.00234246	.00240410
Month.	.00750000	.00770833	.00791666	.00812500

TABLE II.

383. *The Interest of one Pound* per Annum, *at any Rate* per Cent. *from one to ten Pounds, with Halves and Quarters.*

Numb.	1 *per Cent.*	$1\frac{1}{4}$ *per Cent.*	$1\frac{1}{2}$ *per Cent.*	$1\frac{3}{4}$ *per Cent.*
1	0.01000000	0.01250000	0.01500000	0.01750000
2	0.02000000	0.02500000	0.03000000	0.03500000
3	0.03000000	0.03750000	0.04500000	0.05250000
4	0.04000000	0.05000000	0.06000000	0.07000000
5	0.05000000	0.06250000	0.07500000	0.08750000
6	0.06000000	0.07500000	0.09000000	0.10500000
7	0.07000000	0.08750000	0.10500000	0.12250000
8	0.08000000	0.10000000	0.12000000	0.14000000
9	0.09000000	0.11250000	0.13500000	0.15750000

Numb.	2 *per Cent.*	2¼ *per Cent.*	2½ *per Cent.*	2¾ *per Cent.*
1	0.02000000	0.02250000	0.02500000	0.02750000
2	0.04000000	0.04500000	0.05000000	0.05500000
3	0.06000000	0.06750000	0.07500000	0.08250000
4	0.08000000	0.09000000	0.10000000	0.11000000
5	0.10000000	0.11250000	0.12500000	0.13750000
6	0.12000000	0.13500000	0.15000000	0.16500000
7	0.14000000	0.15750000	0.17500000	0.19250000
8	0.16000000	0.18000000	0.20000000	0.22000000
9	0.18000000	0.20250000	0.22500000	0.24750000

Numb.	3 *per Cent.*	3¼ *per Cent.*	3½ *per Cent.*	3¾ *per Cent.*
1	0.03000000	0.03250000	0.03500000	0.03750000
2	0.06000000	0.06500000	0.07000000	0.07500000
3	0.09000000	0.09750000	0.10500000	0.11250000
4	0.12000000	0.13000000	0.14000000	0.15000000
5	0.15000000	0.16250000	0.17500000	0.18750000
6	0.18000000	0.19500000	0.21000000	0.22500000
7	0.21000000	0.22750000	0.24500000	0.26250000
8	0.24000000	0.26000000	0.28000000	0.30000000
9	0.27000000	0.29000000	0.31500000	0.33750000

Years.	4 *per Cent.*	4¼ *per Cent.*	4½ *per Cent.*	4¾ *per Cent.*
1	0.04000000	0.04250000	0.04500000	0.04750000
2	0.08000000	0.08500000	0.09000000	0.09500000
3	0.12000000	0.12750000	0.13500000	0.14250000
4	0.16000000	0.17000000	0.18000000	0.19000000
5	0.20000000	0.21250000	0.22500000	0.23750000
6	0.24000000	0.25500000	0.27000000	0.28500000
7	0.28000000	0.29750000	0.31500000	0.33250000
8	0.32000000	0.34000000	0.36000000	0.38000000
9	0.36000000	0.38250000	0.40500000	0,42750000

Years.

Years.	5 *per Cent.*	$5\frac{1}{4}$ *per Cent.*	$5\frac{1}{2}$ *per Cent.*	$5\frac{3}{4}$ *per Cent.*
1	0.05000000	0.05250000	0.05500000	0.05750000
2	0.10000000	0.10500000	0.11000000	0.11500000
3	0.15000000	0.15750000	0.16500000	0.17250000
4	0.20000000	0.21000000	0.22000000	0.23000000
5	0.25000000	0.26250000	0.27500000	0.28750000
6	0.30000000	0.31500000	0.33000000	0.34500000
7	0.35000000	0.36750000	0.38500000	0.40250000
8	0.40000000	0.42000000	0.44000000	0.46000000
9	0.45000000	0.47250000	0.49500000	0.51750000

Years.	6 *per Cent.*	$6\frac{1}{4}$ *per Cent.*	$6\frac{1}{2}$ *per Cent.*	$6\frac{3}{4}$ *per Cent.*
1	0.06000000	0.06250000	0.06500000	0.06750000
2	0.12000000	0.12500000	0.13000000	0.13500000
3	0.18000000	0.18750000	0.19500000	0.20250000
4	0.24000000	0.25000000	0.26000000	0.27000000
5	0.30000000	0.31250000	0.32500000	0.33750000
6	0.36000000	0.37500000	0.39000000	0.40550000
7	0.42000000	0.43750000	0.45500000	0.47250000
8	0.48000000	0.50000000	0.52000000	0.54000000
9	0.54000000	0.56250000	0.58500000	0.60750000

Years.	7 *per Cent.*	$7\frac{1}{4}$ *per Cent.*	$7\frac{1}{2}$ *per Cent.*	$7\frac{3}{4}$ *per Cent.*
1	0.07000000	0.07250000	0.07500000	0.07750000
2	0.14000000	0.14500000	0.15000000	0.15500000
3	0.21000000	0.21750000	0.22500000	0.23250000
4	0.28000000	0.29000000	0.30000000	0.31000000
5	0.35000000	0.36250000	0.37500000	0.38750000
6	0.42000000	0.43500000	0.45000000	0 46500000
7	0.49000000	0.50750000	0.52500000	0.54250000
8	0.56000000	0.58000000	0.60000000	0.62000000
9	0.63000000	0.65250000	0.67500000	0.69750000

Years.

Years.	8 *per Cent.*	8¼ *per Cent.*	8½ *per Cent.*	8¾ *per Cent.*
1	0.08000000	0.08250000	0.08500000	0.08750000
2	0.16000000	0.16500000	0.17000000	0.17500000
3	0.24000000	0.24750000	0.25500000	0.26250000
4	0.32000000	0.33000000	0.34000000	0.35000000
5	0.40000000	0.41250000	0.42500000	0.43750000
6	0.48000000	0.49500000	0.51000000	0.52500000
7	0.56000000	0.57750000	0.59500000	0.61250000
8	0.64000000	0.66000000	0.68000000	0.70000000
9	0.72000000	0.74250000	0.76500000	0.78750000

Years.	9 *per Cent.*	9¼ *per Cent.*	9½ *per Cent*	9¾ *per Cent.*
1	0.09000000	0.09250000	0.09500000	0.09750000
2	0.18000000	0.18500000	0.19000000	0.19500000
3	0.27000000	0.27750000	0.28500000	0.29250000
4	0.36000000	0.37000000	0.38000000	0.39000000
5	0.45000000	0.46250000	0.47500000	0.48750000
6	0.54000000	0.55500000	0.57000000	0.58500000
7	0.63000000	0.64750000	0.66500000	0.68250000
8	0.72000000	0.74000000	0.76000000	0.78000000
9	0.81000000	0.83250000	0.85500000	0.87750000

TABLE III.

Of SIMPLE INTEREST.

384. *The* Rebate, *or* Discount *of one Pound for* Days, *at the Rates of* 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, 6, per Cent. per Annum.

Days.	2 *per Cent.*	$2\frac{1}{2}$ *per Cent.*	3 *per Cent.*	$3\frac{1}{2}$ *per Cent.*
1	.0000548	.0000685	.0000822	.0000959
2	.0001096	.0001370	.0001644	.0001917
3	.0001644	.0002054	.0002465	.0002876
4	.0002191	.0002739	.0003287	.0003834
5	.0002739	.0003424	.0004108	.0004792
6	.0003287	.0004108	.0004929	.0005750
7	.0003834	.0004792	.0005750	.0006708
8	.0004382	.0005477	.0006571	.0007666
9	.0004929	.0006161	.0007392	.0008623
10	.0005477	.0006845	.0008212	.0009580
20	.0010947	.0013680	.0016411	.0019141
30	.0016411	.0020506	.0024597	.0028685
40	.0021870	.0027322	.0032769	.0038210
50	.0027322	.0034139	.0040928	.0047716
60	.0032769	.0040928	.0049073	.0057205
70	.0038210	.0047716	.0057205	.0066676
80	.0043644	.0054496	.0065324	.0076128
90	.0049073	.0061266	.0073429	.0085563
100	.0054496	.0068027	.0081522	.0094980
110	.0059913	.0074779	.0089601	.0104379
120	.0065324	.0081522	.0097667	.0113760
130	.0070729	.0088255	.0105720	.0123123
140	.0076128	.0094980	.0113760	.0132468
150	.0081522	.0101695	.0121786	.0141796
160	.0086909	.0108401	.0129780	.0151106
170	.0092291	.0115098	.0137801	.0160399
180	.0097667	.0121786	.0145788	.0169674
190	.0103037	.0128465	.0153763	.0178932
200	.0108401	.0135135	.0161725	.0188172
210	.0113759	.0141796	.0169674	.0197395

Table III.

The Discount of one Pound for Days.

Days.	4 per Cent.	$4\frac{1}{2}$ per Cent.	5 per Cent.	6 per Cent.
1	.0001096	.0001233	.0001370	.0001644
2	.0002191	.0002465	.0002739	.0003287
3	.0003287	.0003697	.0004108	.0004929
4	.0004382	.0004929	.0005477	.0006571
5	.0005477	.0006161	.0006845	.0008212
6	.0006571	.0007392	.0008212	.0009853
7	.0007665	.0008623	.0009580	.0011494
8	.0008759	.0009853	.0010947	.0013133
9	.0009853	.0011084	.0012314	.0014773
10	.0010947	.0012314	.0013680	.0016411
20	.0021870	.0024597	.0027322	.0032769
30	.0032769	.0036850	.0040928	.0049073
40	.0043644	.0049073	.0054496	.0065324
50	.0054496	.0061266	.0068027	.0081522
60	.0065234	.0073429	.0081522	.0097667
70	.0076128	.0085563	.0094980	.0113760
80	.0086909	.0097667	.0108401	.0129780
90	.0097667	.0109741	.0121786	.0145788
100	.0108401	.0121786	.0135135	.0161725
110	.0119112	.0133802	.0148448	.0177610
120	.0129800	.0145788	.0161725	.0193444
130	.0140465	.0157746	.0174966	.0209228
140	.0151006	.0169674	.0188172	.0224960
150	.0161725	.0181574	.0201342	.0240642
160	.0172321	.0193444	.0214477	.0256273
170	.0182894	.0205286	.0227577	.0271855
180	.0193444	.0217100	.0240642	.0287387
190	.0203972	.0228885	.0253672	.0302869
200	.0214477	.0240642	.0266667	.0318302
210	.0224960	.0252370	.0279627	.0333686

TABLE III.

The Discount of one Pound for Days.

Days.	2 *per Cent.*	$2\frac{1}{2}$ *per Cent.*	3 *per Cent.*	$3\frac{1}{2}$ *per Cent.*
220	.0119112	.0148448	.0177610	.0206601
230	.0124459	.0155091	.0185534	.0215789
240	.0129800	.0161725	.0193444	.0224959
250	.0135135	.0168350	.0201342	.0234114
260	.0140465	.0174966	.0209227	.0243251
270	.0145788	.0181574	.0217100	.0252370
280	.0151106	.0188172	.0224960	.0261473
290	.0156418	.0194762	.0232807	.0270558
300	.0161725	.0201342	.0240642	.0279627
310	.0167026	.0207914	.0248464	.0288679
320	.0172321	.0214477	.0256273	.0297714
330	.0177610	.0221031	.0264070	.0306732
340	.0182894	.0227577	.0271855	.0315734
350	.0188172	.0234114	.0279627	.0324718
360	.0193444	.0240642	.0287387	.0333686
361	.0193971	.0241294	.0288162	.0334582
362	.0194498	.0241946	.0288937	.0335478
363	.0195025	.0242598	.0289712	.0336374
364	.0195552	.0243251	.0290487	.0337269
365	.0196078	.0243902	.0291262	.0338164

TABLE III.

The Discount of one Pound for Days.

Days.	4 *per Cent.*	4½ *per Cent.*	5 *per Cent.*	6 *per Cent.*
220	.0235420	.0264070	.0292553	.0349022
230	.0245858	.0275743	.0305445	.0364309
240	.0256273	.0287387	.0318302	.0379547
250	.0266667	.0299003	.0331126	.0394737
260	.0277038	.0310592	.0343915	.0409879
270	.0287387	.0322153	.0356671	.0424974
280	.0297714	.0333686	.0369393	.0440021
290	.0308019	.0345192	.0382082	.0455021
300	.0318302	.0356671	.0394737	.0469974
310	.0328564	.0368122	.0407352	.0484880
320	.0338804	.0379547	.0419948	.0499740
330	.0349022	.0390444	.0432503	.0514553
340	.0359218	.0402314	.0445026	.0529320
350	.0369393	.0413657	.0457516	.0544041
360	.0379547	.0424974	.0469974	.0558717
361	.0380561	.0426104	.0471218	.0560182
362	.0381575	.0427234	.0472462	.0561647
363	.0382588	.0428364	.0473705	.0563111
364	.0383602	.0429493	.0474948	.0564575
365	.0384615	.0430622	.0476191	.0566038

CHAP.

CHAP. XIX.

Of Compound Interest, Annuities, *&c.*

385. Compound Intereſt is that which ariſes from any Principal and its Intereſt put together, as the Intereſt ſo becomes due, ſo that at every Payment, or at the Time when the Payments became due, there is created a new Principal; and for that Reaſon it is called Intereſt upon Intereſt, or Compound Intereſt. As for Example: Suppoſe 100*l.* were lent out for two Years at 6*l.* *per Cent. per Ann.* Compound Intereſt; then at the End of the firſt Year it will only amount to 106*l.* as in Simple Intereſt; but for the ſecond Year this 106*l.* becomes Principal, which will amount to 112*l.* 7*s.* $2\frac{1}{2}$*d.* at the ſecond Years End, whereas by Simple Intereſt would have amounted to but 112*l.* And though it be not lawful to let out Money at Compound Intereſt, yet in purchaſing of Annuities or Penſions, *&c.* and taking Leaſes in Reverſion, it is very uſual to allow Compound Intereſt to the Purchaſer for his ready Money; and therefore it is requiſite to underſtand it.

386. Now in order to raiſe the Theorems for Compound Intereſt, we muſt conſider that the Amount of 1*l.* and its Intereſt, for one Year, is here called the *Ratio*, and is found by the following Proportion for any Rate of Intereſt. Thus,

$$\text{As}\begin{cases} \overset{\pounds.}{100} : 105 :: 1 : 1,05 = \text{R, at 5 } \textit{per Cent.} \\ 100 : 106 :: 1 : 1,06 = \text{R, at 6 } \textit{per Cent,} \text{ \&c.} \end{cases}$$

387. But as one Pound is to the Amount of one Pound at one Year's End, ſo is that Amount to the Amount of one Pound at two Years End; and ſo on continually.

That is, $1 : R :: R : R :: RR : RRR :: R^3 : R^4 :: R^4 :: R^5 ::$ *&c.*

$$\text{Then}\begin{cases} 1. \quad 2. \quad 3. \quad 4. \quad 5. = \text{Years.} \\ R.\ R^2.\ R^3.\ R^4.\ R^5 = \text{the Amount of 1}\textit{l.}\text{ at any Rate.} \end{cases}$$

388. Hence 'tis evident the Amount proceeds in a geometrical Proportion, wherein the Time ($= t$) or Number of Years, is always to or the ſame with the Index of the Power of the laſt and higheſt Term of the Series, *viz.* R^5 or R^t; for in Compound Intereſt the Letters denote as below.

$P =$ the Principal put to Intereſt.
$t =$ the Time of its Continuance.
$A =$ the Amount of the Principal and Intereſt.
$R =$ the Amount of 1*l.* and its Intereſt for one Year, at any given Rate, which may be thus found.

389. Now as one Pound is to the Amount of one Pound for any given Time, (t) ſo is any propoſed Principal or Sum to its Amount for the ſame Time.

That is, As $1 : R^t :: P : PR^t$.
But $PR^t = A$, the general Theorem.

From this Theorem any three of the four Quantities P, R, t, A, being given, the other may be found. But becauſe t in all theſe Caſes is the Index, or denotes the Power to which the Quantity R muſt be raiſed, and is in itſelf uſually large, it plainly appears that in the Solution of this Sort of Queſtions, there will be great Labour required, if we proceed in the common algebraic Way of extracting of Roots, or raiſing of Powers, which therefore we ſhall here decline, and ſhew how this may be done very eaſily by Logarithms, and alſo by Tables ready computed, for ſhewing the Value of R^t for any Number of Days and Years, and at any of the uſual Rates of Intereſt: And firſt by Logarithms.

390. From the Theorem $PR^t = A$, we have $R^t = \frac{A}{P}$ which muſt now be expreſſed logarithmically; thus, $R^t = t \times$ L. R, (here L denotes the Logarithm of the Quantity to which it is prefixed. See 134, 141.) and $\frac{A}{P} =$ L. A — L. P; and ſo $t \times$ L. R $=$ L. A — L. P; whence we have the following Theorems.

$t \times$ L

$t \times L.R + L.P = L.A$, Theorem I.

$L.A - t \times L.R = L.P$, Theorem II.

$\frac{L.A - L.P}{t} = L.R$, Theorem III.

$\frac{L.A - L.P}{L.R} = t$, Theorem IV.

391. We shall illustrate these Theorems by the following Examples.

QUESTION I.

What will 275*l.* 15*s.* amount to in three Years and an half, at $4\frac{1}{2}$ *per Cent. per Ann.* Compound Interest?

Here $\begin{cases} P = 275,75, \text{ the Principal.} \\ R = 1,045, \text{ the Rate of Interest.} \\ t = 3,5, \text{ the Time.} \\ \text{To find A, the Amount, } per \text{ Theor. I.} \end{cases}$

The Logarithm of the Rate . . R = 1,045 = 0,019116
Multiply by the Time . . . . t = 3,5

The Product is the Logarithm of R^t = 1,1665 = 066906
To which add the Logarithm of P = 275,75 = 2,440515

The Sum is the Log. of PR^t = L. A = 321,68 = 2,507421

So the Amount sought is 321*l.* 13*s.* 7*d.* which is more than the Amount by Simple Interest by 2*l.* 10*s.* as may be found in Theorem I. Article 270.

QUESTION II.

392. What Principal or Sum being put to Use at $4\frac{1}{2}$ *per Cent.* Compound Interest, will amount to 321*l.* 13*s.* 7*d.* in three Years and an half?

Here $\begin{cases} P = 321,68, \text{ the Principal.} \\ R = 1,045 \text{ the Rates of Interest.} \\ t = 3,5, \text{ the Time.} \\ \text{To find A, the Amount, } per \text{ Theor. II.} \end{cases}$

From

From the Logarithm of . . . $A = 321,68 = 2,507421$
Subduct the Logarithm of . . $R^t = 1,1665 = 0,066906$

The Differ. is the Log. of $\frac{A}{R^t} = L.P = 275,75 = 2,440515$

Therefore the Principal sought is 275*l.* 15*s.*

Question III.

393. At what Rate *per Cent.* &c. Compound Interest, will 275*l.* 15*s.* raise a Stock, or amount to 321*l.* 13*s.* 7*d.* in three Years and an Half?

$$\text{Here}\begin{cases} P = 321,68. \\ R = 275,75 \\ t = 3,5. \\ \text{To find A, the Amount, } per \text{ Theor. III.} \end{cases}$$

From the Logarithm of . . . $A = 321,68 = 2,50742$
Subtract the Logarithm of . . $P = 275,75 = 2,440515$

The Difference is the Logarithm of R^t . . . . $0,066906$
Which divide by the Time . . . . $t = 3,5$

The Quotient is the Logarithm of $R = 1,045 = 0,019116$
Then as $1 : 0,045 :: 100 : 4,5 = 4\frac{1}{2}$*l.* *per Cent.* the Rate required.

Question IV.

394. In what Time will 275*l.* 15*s.* raise a Stock of 321*l.* 13*s.* 7*d.* at the Rate of $4\frac{1}{2}$*l.* *per Cent.* Compound Interest?

$$\text{Here}\begin{cases} P = 275,75. \\ R = 321,68. \\ t = 1,045. \\ \text{To find A, the Amount, } per \text{ Theor. IV.} \end{cases}$$

From the Logarithm of . . $A = 321,68 = 2,507421$
Subduct the Logarithm of . . . $P = 275,75 = 2,440515$

The Difference is the Logarith. of $R^t = 1,045^t = 0,066906$

Then

Then the Logar. of 1,045 = 0,019116) 0,066906 (3,5 = t

	57348
	95580
	95580
	.....

Thus the Anſwer is three Years and ſix Months.

Note, As the Amount of any Sum and its Intereſt is greater at Compound Intereſt than at Simple Intereſt, for any Time above a Year, ſo it is leſs at Compound than at Simple Intereſt for any Time leſs than a Year, as the Learner may eaſily prove by the Theorems before going.

395. From the above Examples it appears how expeditiouſly Queſtions of Compound Intereſt are ſolved by Logarithms; but as few People concerned in this Affair underſtand this uſeful Method of Computation, we ſhall next exhibit a Set of Tables, wherein the Value of R^t is expreſſed for any Time, or Rate of Intereſt it can be required, and then the whole Buſineſs of Compound Intereſt is done by Multiplication and Diviſion only.

396. The firſt Table expreſſes R^t for Days, and is made from the general Theorem $PR^t = A$; thus, Put $P = 1l.$ then will the Theorem be $R^t = A$: Suppoſe $R = 1,05l.$ for one Year, or 365 Days, then we have $\sqrt[365]{R} = A =$ Amount for one Day. This 365th Root of R may be extracted algebraically, or be found by a large Table of Logarithms, near enough for Uſe: And if $R = 1,05l.$ then $\sqrt[365]{1,05} = 1,0003368l.$ = the Amount of 1*l.* for one Day, at the Rate of 5 *per Cent.* Compound Intereſt.

397. Then ſay: As 1 : 1,00013368 : : 1,00013368 : 1,00026738 = the Amount of 1*l.* for two Days, and ſo on; or thus,

$$
\begin{array}{rll}
\text{If } R^1 = & 1,00013368 = & \text{Amount of } 1l. \text{ for } 1 \text{ Day.} \\
\text{Multiply by } R = & 1,00013368 & \\
\hline
\text{The Product is } R^2 = & 1,00026738 = & \text{Amount for two Days.} \\
\text{Multiply again by } R = & 1,00013368 & \\
\hline
\text{The Product is } R^3 = & 1,00040110 = & \text{Amount for three Days.}
\end{array}
$$

And

And thus the Amounts for the Days, at the ſeveral Rates of Intereſt in the firſt Table, are found.

398. In the ſame Manner is the ſecond Table conſtructed for Years; for if $R^t = A = 1{,}05\,l.$ then ſay as before $1 : 1{,}05 :: 1{,}05 : 1{,}1025 = R^1$ = Amount of 1*l.* at the End of the ſecond Year; then $1{,}1025 \times 1{,}05 = 1{,}157625 = R^3$ = Amount of 1*l.* at the End of the third Year; and thus you proceed for the Amounts at the End of every other Year in the Table.

399. Now the Uſe of theſe Tables is extremely eaſy; for let N be put for any Number in the Table; that is, let $R^t = N$, then the Theorem $PR^t = A$, will be $PN = A$; whence we have $P = \frac{A}{N}$; and ſo the *Amount of any Principal is known by multiplying the ſaid Principal by the tabular Number for the given Rate and Time*; and the Amount divided by the ſaid Number gives the Principal.

EXAMPLE.

400. What will 246*l.* amount to in 30 Days, or Years, at 5 *per Centum?*

In Table I. againſt 30 Days, under 5 *per Cent.* ſtands	1,0040182
Which multiplied by the Principal	246
The Product is the Amount required, *viz.*	£. 247,06847

In Table II. againſt 30 Years, and 5 *per Cent.* is	4,3219424
Which multiplied by the Principal	246
Gives the Amount for that Time, *viz.*	£. 1063,1978, &c.

401. By the Reverſe the Principal is found for any given Amount, which needs no Example. If it happens that the Amount be required for any Number of Days or Years that are not in the Tables, then obſerve this

RULE.

RULE.

Divide the given Number of Days, or Years, into two ſuch Numbers as are in the Table, then multiply the Amounts pertaining to each, into each other, the Product will be the Amount for the Time required.

EXAMPLE I.

What will 523*l.* amount to, in 194 Days, at 5 *per Cent. per Annum?*

The two Parts of this Number in the Table are 190 and 4; therefore,

In Table I. againſt 190 Days, under 5 *per Cent.* is	1.0257228
And againſt 4 Days, at the ſame Rate, is - -	1.0005348
The Product is the Amount of 1*l.* for 194 Days, *viz.* - - - - -	1.0262714
Which multiply by the principal Sum, *viz.* - -	523
This Product is the Anſwer - - *l.*	536.7399840

In Money, 536*l.* 14*s.* $9\frac{1}{2}$*d.*

EXAMPLE II.

What is the Amount of 150*l.* in 91 Years, at 5 *per Cent.?*

In Table II. againſt 50 Years, under 5 *per Cent.* is	11.4674000
And againſt 41 Years, at 5 *per Cent.* is - -	7.3919881
The Product is the Amount of 1*l.* for 91 Years, *viz.* - - - -	84.7668833
Which multiply by the principal Sum, *viz.* - -	150
The Product is the Anſwer - - *l.*	12715.032495

In Money, 12715*l.* 0*s.* $7\frac{1}{2}$*d.*

EXAMPLE. III.

What will 523*l.* amount to in 5 Years and 194 Days, at 5 *per Cent?*

In Table II. against 5 Years, at 5 *per Cent.* is	1.2762816
And the Amount of 1*l.* in 194 Days, as above, is	1.0262714
The Product is the Amount of 1*l.* in 5 Years and 194 Days - - -	1.3098113
Which multiplied by the principal Sum - -	523
The Product is the Answer, *viz.* - -	*l.* 685.0313413

In Money, 685*l.* 0*s.* $7\frac{1}{2}$*d.*

N. B. The Reason of this Rule is, that the Numbers of Days and Years are in arithmetrical Progression, and the correspondent Numbers in the Tables are in geometrical Progression, (396, 397.) and therefore to the *Addition* or *Sum* of any two of the former, there answers a *Multiplication*, or a *Product* of the latter. (See 132.)

TABLES

TABLES of COMPOUND INTEREST.

TABLE I.

The Amount *of one Pound for* Days, *at the Rates of* 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, *and* 6, per Cent. per Annum.

Days.	*2 per Cent.*	*$2\frac{1}{2}$ per Cent.*	*3 per Cent.*	*$3\frac{1}{2}$ per Cent.*
1	1.0000542	1.0000676	1.0000809	1.0000942
2	1.0001085	1.0001353	1.0001619	1.0001885
3	1.0001627	1.0002029	1.0002429	1.0002827
4	1.0002170	1.0002706	1.0003240	1.0003770
5	1.0002713	1.0003383	1.0004050	1.0004713
6	1.0003255	1.0004059	1.0004860	1.0005656
7	1.0003798	1.0004736	1.0005670	1.0006600
8	1.0004341	1.0005412	1.0006480	1.0007542
9	1.0004884	1.0006090	1.0007291	1.0008486
10	1.0005426	1.0006767	1.0008101	1.0009429
20	1.0010856	1.0013539	1.0016209	1.0018867
30	1.0016289	1.0020315	1.0024324	1.0028315
40	1.0021725	1.0027097	1.0032445	1.0037771
50	1.0027163	1.0033882	1.0040573	1.0047236
60	1.0032605	1.0040673	1.0048708	1.0056710
70	1.0038049	1.0047468	1.0056849	1.0066193
80	1.0043497	1.0054267	1.0064996	1.0075685
90	1.0048947	1.0061071	1.0073151	1.0085186
100	1.0054401	1.0067880	1.0081311	1.0094696
110	1.0059857	1.0074693	1.0089479	1.0104214
120	1.0065316	1.0081511	1.0097653	1.0113742
130	1.0070779	1.0088334	1.0105834	1 0123279
140	1.0076244	1.0095161	1.0114021	1.0132825
150	1.0081712	1.0101993	1.0122215	1.0142379
160	1.0087183	1.0108829	1.0130415	1.0151943

TABLES of Compound Interest.

TABLE I.

The Amount of one Pound, Compound-Interest.

Days.	4 *per Cent.*	$4\frac{1}{2}$ *per Cent.*	5 *per Cent.*	6 *per Cent.*
1	1.0001074	1.0001206	1.0001336	1.0001596
2	1.0002140	1.0002412	1.0002673	1.0003193
3	1.0003224	1.0003618	1.0004011	1.0004799
4	1.0004299	1.0004824	1.0005348	1.0006387
5	1.0005374	1.0006031	1.0006685	1.0007985
6	1.0006449	1.0007238	1.0008023	1.0009583
7	1.0007524	1.0008445	1.0009361	1.0011181
8	1.0008600	1.0009652	1.0010699	1.0012779
9	1.0009675	1.0010859	1.0012037	1.0014378
10	1.0010751	1.0012066	1.0013376	1.0015976
20	1.0021513	1.0024148	1.0026770	1.0031979
30	1.0032288	1.0036243	1.0040182	1.0048007
40	1.0043074	1.0048354	1.0053611	1.0064060
50	1.0053871	1.0060479	1.0067059	1.0080139
60	1.0064680	1.0072618	1.0080525	1.0096244
70	1.0075501	1.0084773	1.0094009	1.0112375
80	1.0086333	1.0096942	1.0107511	1.0128531
90	1.0097177	1.0109125	1.0121031	1.0144713
100	1.0108033	1.0121324	1.0134569	1.0160921
110	1.0118900	1.0133537	1.0148125	1.0177155
120	1.0129779	1.0145765	1.0161699	1.0193415
130	1.0140670	1.0158007	1.0175291	1.0209701
140	1.0151572	1.0170265	1.0188902	1.0226013
150	1.0162487	1.0182537	1.0202531	1.0242351
160	1.0173412	1.0194824	1.0216178	1.0258715

TABLE I.

The Amount of one Pound, Compound-Interest.

Days	2 *per C.*	$2\frac{1}{2}$ *per C.*	3 *per C.*	$3\frac{1}{2}$ *per C.*
170	1.0092658	1.0115670	1.0138623	1.0161516
180	1.0098135	1.0122516	1.0146837	1.0171098
190	1.0103615	1.0129366	1.0155057	1.0180689
200	1.0109098	1.0136221	1.0163284	1.0190288
210	1.0114584	1.0143081	1.0171518	1.0199897
220	1.0120073	1.0149945	1.0179759	1.0209515
230	1.0125565	1.0154814	1.0188006	1.0219142
240	1.0131060	1.0163687	1.0196260	1.0228778
250	1.0136558	1.0170565	1.0204520	1.0238424
260	1.0142059	1.0177448	1.0212788	1.0248078
270	1.0147563	1.0184336	1.0221062	1.0257741
280	1.0153070	1.0191228	1.0229342	1.0267414
290	1.0158580	1.0198125	1.0237630	1.0277096
300	1.0164093	1.0205026	1.0245924	1.0286786
310	1.0169609	1.0211932	1.0254225	1.0296486
320	1.0175127	1.0218843	1.0262532	1.0306195
330	1.0180649	1.0225758	1.0270847	1.0315914
340	1.0186174	1.0232679	1.0279168	1.0325641
350	1.0191702	1.0239603	1.0287495	1.0335378
360	1.0197233	1.0246533	1.0295830	1.0345123
361	1.0197786	1.0247226	1.0296664	1.0346098
362	1.0198340	1.0247919	1.0297497	1.0347073
363	1.0198893	1.0248613	1.0298331	1.0348049
364	1.0199446	1.0249306	1.0299165	1.0349024
365	1.0200000	1.0250000	1.0300000	1.0350000

TABLE I.

The Amount of one Pound, Compound-Interest.

Days.	4 *per Cent.*	4½ *per Cent.*	5 *per C.*	6 *per C.*
170	1.0184350	1.0207126	1.0229843	1.0275105
180	1.0195299	1.0219442	1.0243527	1.0291522
190	1.0206261	1.0231774	1.0257228	1.0307964
200	1.0217233	1.0244120	1.0270949	1.0324433
210	1.0228218	1.0256481	1.0284687	1.0340928
220	1.0239215	1.0268858	1.0298444	1.0357450
230	1.0250233	1.0281249	1.0312219	1.0373998
240	1.0261243	1.0293655	1.0326013	1.0390572
250	1.0272275	1.0306076	1.0339825	1.0407174
260	1.0283319	1.0318512	1.0353656	1.0423800
270	1.0294375	1.0330963	1.0367505	1.0440454
280	1.0305443	1.0343429	1.0381373	1.0457135
290	1.0316522	1.0355918	1.0395259	1.0473842
300	1.0327614	1.0368406	1.0409164	1.0490576
310	1.0338717	1.0383917	1.0423087	1.0507336
320	1.0349832	1.0393444	1.0437029	1.0524124
330	1.0360960	1.0405985	1.0450990	1.0540938
340	1.0372099	1.0418542	1.0464969	1.0557779
350	1.0383250	1.0431114	1.0478967	1.0574647
360	1.0394413	1.0443700	1.0492984	1.0591542
361	1.0395530	1.0444960	1.0494387	1.0593233
362	1.0396648	1.0446220	1.0495790	1.0594924
363	1.0397765	1.0447479	1.0497193	1.0596616
364	1.0398882	1.0448739	1.0498596	1.0598308
365	1.0400000	1.0450000	1.0500000	1.0600000

TABLE

TABLE II.

Of Compound Interest,

The Amount *of one Pound for* Years, *at the Rates of* 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, *and* 6, per Cent, per Annum.

Years.	*2 per Cent.*	*$2\frac{1}{2}$ per Cent.*	*3 per Cent.*	*$3\frac{1}{2}$ per Cent.*
1	1.0200000	1.0250000	1.0300000	1.0350000
2	1.0404000	1.0506250	1.0609000	1.0712250
3	1.0612080	1.0768906	1.0927270	1.1087178
4	1.0824321	1.1038128	1.1255088	1.1475230
5	1.1040808	1.1314082	1.1592740	1.1876863
6	1.1261624	1.1596934	1.1948523	1.2292553
7	1.1486856	1.1886857	1.2298733	1.2722792
8	1.1716593	1.2184029	1.2667700	1.3168098
9	1.1950925	1.2488629	1.3047731	1.3628973
10	1.2189944	1.2800845	1.3439163	1.4105987
11	1.2433743	1.3120366	1.3842338	1.4599697
12	1.2682417	1.3448888	1.4257608	1.5110686
13	1.2936066	1.3785110	1.4685337	1.5639560
14	1.3194787	1.4129738	1.5125897	1.6186945
15	1.3458683	1.4482981	1.5579654	1.6753488
16	1.3727857	1.4845056	1.6047064	1.7339860
17	1.4002414	1.5216182	1.6528476	1.7946755
18	1.4282462	1.5596587	1.7024330	1.8574892
19	1.4568111	1.5986501	1.7535060	1.9225013
20	1.4859474	1.6386164	1.8061112	1.9897888
21	1.5156663	1.6795818	1.8602945	2.0594314
22	1.5459796	1.7215714	1.9161034	2.1315115
23	1.5768992	1.7646106	1.9735865	2.2061144
24	1.6084372	1.8087259	2.0327941	2.2833284
25	1.6406059	1.8539441	2.0937779	2.3632449

TABLE

TABLE II.

The Amount of one Pound, Compound-Interest.

Years.	4 *per Cent*	4½ *per Cent.*	5 *per Cent.*	6 *per Cent.*
1	1.0400000	1.0450000	1.0500000	1.0600000
2	1.0816000	1.0920250	1.1025000	1.1236000
3	1.1248640	1.1411661	1.1576250	1.1910160
4	1.1698586	1.1925186	1.2155063	1.2624769
5	1.2166529	1.2461819	1.2762816	1.3382256
6	1.2653190	1.3022601	1.3400956	1.4185191
7	1.3159318	1.3608618	1.4071004	1.5036303
8	1.3685691	1.4221006	1.4774554	1.5938481
9	1.4233118	1.4860951	1.5513282	1.6894703
10	1.4802443	1.5529694	1.6288946	1.7908477
11	1.5394541	1.6228530	1.7103393	1.8982980
12	1.6010322	1.6958814	1.7958563	2.0121965
13	1.6650735	1.7721961	1.8156491	2.1329283
14	1.7316764	1.8519449	1.9799316	2.2609039
15	1.8009435	1.9352824	2.0789282	2.3965582
16	1.8729812	2.0223701	2.1828746	2.5402517
17	1.9479005	2.1133768	2.2920113	2.6927728
18	2.0258165	2.2084787	2.4066192	2.8543392
19	2.1068492	2.3078603	2.5269502	3.0255995
20	2.1911231	2.4117149	2.6532977	3.2071355
21	2.2787681	2.5202111	2.7859626	3.3995636
22	2.3699188	2.6336520	2.9252607	3.6035374
23	2.4647155	2.7521663	3.0715238	3.8197497
24	2.5433042	2.8760838	3.2251000	4.0489346
25	2.6658363	3.0054344	3.3863549	4.2918707

TABLE

TABLE II.

The Amount of one Pound, Compound-Interest.

Years.	2 *per Cent.*	$2\frac{1}{2}$ *per Cent.*	3 *per Cent.*	$3\frac{1}{2}$ *per Cent.*
26	1.6734181	1.9002927	2.1565912	2.4459585
27	1.7068864	1.9478000	2.2212890	2.5315871
28	1.7410242	1.9964950	2.2879276	2.6201719
29	1.7758446	2.0464073	2.3565655	2.7118779
30	1.8113615	2.0975675	2.4272624	2.8067937
31	1.8475888	2.1500067	2.5000803	2.9050314
32	1.8845405	2.2037569	2.5750827	3.0067075
33	1.9222314	2.2588508	2.6523352	3.1119423
34	1.9606760	2.3153221	2.7319053	3.2208603
35	1.9998895	2.3732051	2.8138624	3.3335904
36	2.0398873	2.4325353	2.8982783	3.4502661
37	2.0806850	2.4933487	2.9852266	3.5710254
38	2.1222987	2.5556824	3.0747834	3.6960113
39	2.1647447	2.6195744	3.1670269	3.8253717
40	2.2080396	2.6850638	3.2620377	3.9592597
41	2.2522004	2.7521904	3.3598989	4.0978338
42	2.2979444	2.8209957	3.4606958	4.2412579
43	2.3431893	2.8915500	3.5645167	4.3897020
44	2.3900531	2.9638080	3.6714522	4.5433416
45	2.4378542	3.0379032	3.7815958	4.7023585
46	2.4866112	3.1138508	3.8950437	4.8669411
47	2.5343435	3.1916971	4.0118950	5.0372840
48	2.5870703	3.2714895	4.1322518	5.2135889
49	2.6388117	3.3532768	4.2562194	5.3960645
50	2.6915880	3.4371087	4.3839060	5.5849268

TABLE II.

The Amount of one Pound, Compound-Interest.

Years.	4 *per Cent.*	4 ½ *perCent.*	5 *per Cent.*	6 *per Cent.*
26	2.7724697	3.1406790	3.5556727	4.5493829
27	2.8833685	3.2820095	3.7334563	4.8223459
28	2.9987033	3.4296999	3.9201291	5.1116866
29	3.1186514	3.5840364	4.1161356	5.4183878
30	3.2433975	3.7453181	4.3219424	5.7434911
31	3.3731334	3.9138574	4.5380395	6.0881006
32	3.5080587	4.0899810	4.7649415	6.4533866
33	3.6483811	4.2740301	5.0031885	6.8405898
34	3.7943163	4.4663615	5.2533480	7.2510252
35	3.9460889	4.6673478	5.5160154	7.6860869
36	4.1039325	4.8773784	5.7918161	8.1472519
37	4.2680898	5.0968604	6.0814069	8.6360870
38	4.4388134	5.3262192	6.3854773	9.1542523
39	4.6163659	5.5658990	6.7047511	9.7035074
40	4.8010206	5.8163645	7.0399887	10.2857178
41	4.9930614	6.0781009	7.3919881	10.9028609
42	5.1927839	6.3516154	7.7615875	11.5570326
43	5.4004952	6.6374381	8.1496669	12.2504545
44	5.6165150	6.9361229	8.5571503	12.9854818
45	5.8411756	7.2482484	8.9850078	13.7646107
46	6.0748227	7.5744196	9.4342582	14.5904873
47	6.3178156	7.9152684	9.9059711	15.4659166
48	6.5705282	8.2714555	10.4012696	16.3938716
49	6.8333493	8.6436710	10.9213331	17.3775039
50	7.1066833	9.0326362	11.4674000	18.4201541

402 The

402. The Construction of Table III. is by Theorem II. *viz.* $\frac{A}{R^t} = P$, the present Worth or Value of one Pound, which is here to be considered as the Amount; therefore if $A = 1$ *l.* $R = 1{,}05$, and $t = 1, 2, 3, 4, 5$, and Years, as before; 'tis evident that Unity, or 1, being divided by the Numbers in the second Table (designed by R^t) will give the Numbers in this third Table, or the present Values of 1*l.* for the *Tabular Years*, and 5 *per Cent.* and so for any other Rate of Interest.

Example at 5 per Cent.

Unity or 1 divided by	1,05	The Quot. is	,952381	The present Value of 1*l.* for	1	Years, &c.
	1,1025		,907029		2	
	1,157625		,863837		3	
	1,21550625		,822702		4	
	1,276281		,783526		5	

403. The Use of the Table is very easy, as will appear from the following Example.

What is the present Worth of 246*l.* due at the End of 30 Years, allowing 5 *per Cent.* Compound Interest?

In Table III. against the given Time and Rate, is the Number		0,2313775
Which multiply by the given Sum	- -	246
The Product is the *present Worth* required	£.	56,9189, &c.

404. Note, in this Table you have the Numbers for whole Years only; but if the Time consists of Years, and Parts of a Year, you take the Difference between the two contiguous whole Years, and then a proportional Part of that Difference subducted from the tabular Number of the given Years, will be the tabular Number required for the given Time.

TABLE III.

Of COMPOUND INTEREST.

405. *The* present Worth *of one Pound for* Years, *at the Rates of* 2, 2½, 3, 3½, 4, 4½, 5, *and* 6, per Cent. per Annum.

Years.	2 *per C.*	2 ½ *per C.*	3 *per C.*	3½ *per C.*
1	.9803921	.9756097	.9708738	.9661836
2	.9611687	.9518144	.9425959	.9335107
3	.9423223	.9285994	.9151417	.9019427
4	.9238454	.9059506	.8884870	.8714422
5	.9057308	.8838542	.8626088	.8419732
6	.8879713	.8622968	.8374843	.8135006
7	.8705601	.8412653	.8130915	.7859910
8	.8534903	.8207465	.7894092	.7594116
9	.8367552	.8007283	.7664167	.7337710
10	.8203483	.7811984	.7440939	.7089188
11	.8042630	.7621447	.7224213	.6849457
12	.7884931	.7435558	.7013799	.6617833
13	.7730325	.7254203	.6809513	.6394041
14	.7578750	.7077272	.6611178	.6177818
15	.7430147	.6904655	.6418619	.5968906
16	.7284458	.6736249	.6231669	.5767059
17	.7141625	.6571950	.6050164	.5572038
18	.7001593	.6411659	.5873946	.5383611
19	.6864307	.6255277	.5702860	.5201557
20	.6729713	.6102709	.5536758	.5025659
21	.6597758	.5953862	.5375493	.4855709
22	.6468390	.5808646	.5218925	.4691506
23	.6341559	.5666972	.5066917	.4532856
24	.6217214	.5528753	.4919337	.4379571
25	.6095308	.5393905	.4776056	.4231470

TABLE III.

The present Worth of one Pound, Compound-Interest.

Years.	4 *per* C.	4½ *per* C.	5 *per* C.	6 *per* C.
1	.9615385	.9569378	.9523810	.9433962
2	.9245562	.9157299	.9070295	.8899964
3	.8889964	.8762966	.8638376	.8396193
4	.8548042	.8385613	.8227025	.7920937
5	.8219281	.8024511	.7835262	.7472582
6	.7903145	.7618957	.7462154	.7049605
7	.7599178	.7348285	.7106813	.6650571
8	.7306902	.7031851	.6768394	.6274124
9	.7025867	.6729044	.6446089	.5918985
10	.6755642	.6439277	.6139133	.5583948
11	.6495809	.6161987	.5846793	.5267875
12	.6245971	.5896639	.5568374	.4969694
13	.6005741	.5642716	.5303214	.4688390
14	.5774751	.5399729	.5050679	.4423010
15	.5552645	.5167204	.4810171	.4172651
16	.5339082	.4944693	.4581115	.3936463
17	.5133733	.4731764	.4362967	.3713644
18	.4936281	.4528004	.4155207	.3503438
19	.4746424	.4333018	.3957340	.3305130
20	.4563870	.4146429	.3768895	.3118047
21	.4388336	.3967874	.3589424	.2941554
22	.4219554	.3797009	.3418499	.2775051
23	.4057263	.3633501	.3255713	.2617973
24	.3901215	.3477035	.3100679	.2469786
25	.3751168	.3327306	.2953028	.2329986

TABLE

TABLE III.

The present Worth of one Pound, Compound-Interest.

Years.	2 per C.	2 ½ per C.	3 per C.	3 ½ per C.
26	.5975793	.5262347	.4636947	.4088378
27	.5858620	.5133997	.4501891	.3950123
28	.5743746	.5008778	.4370768	.3816543
29	.5631123	.4886613	.4243464	.3687482
30	.5520709	.4767427	.4119868	.3562784
31	.5412460	.4651148	.3999871	.3442304
32	.5306333	.4537706	.3883370	.3325897
33	.5202287	.4427030	.3770263	.3213427
34	.5100282	.4319053	.3660499	.3104761
35	.5000276	.4213711	.3553834	.2999769
36	.4902232	.4110937	.3450324	.2898327
37	.4806109	.4010671	.3349829	.2800316
38	.4711872	.3912849	.3252262	.2705619
39	.4619482	.3817414	.3157536	.2614125
40	.4528904	.3724306	.3065568	.2515725
41	.4440102	.3633470	.2976280	.2440314
42	.4303041	.3544848	.2889592	.2357791
43	.4267608	.3458389	.2805429	.2278059
44	.4184008	.3374038	.2723716	.2201023
45	.4101968	.3291744	.2644386	.2126594
46	.4021537	.3211458	.2567395	.2054679
47	.3942684	.3133129	.2492588	.1985197
48	.3865376	.3056712	.2419988	.1918065
49	.3789584	.2982158	.2349503	.1853202
50	.3715279	.2909422	.2281071	.1790534

TABLE III.

The present Worth of one Pound, Compound-Interest.

Years.	4 per C.	4 ½ per C.	5 per C.	6 per C.
26	.3606892	.3184025	.2812407	.2198100
27	.3468166	.3046914	.2678483	2073680
28	.3334775	.2915707	.2550936	.1956301
29	.3206514	.2790150	.2429463	.1845567
30	.3083187	.2670000	.2313775	.1741101
31	.2964603	.2555024	.2203595	.1642548
32	.2850579	.2444999	.2098662	.1549574
33	.2740942	.2339712	.1998762	.1461862
34	.2635521	2238959	.1903548	.1379115
35	.2534155	.2142544	.1812903	.1301052
36	.2436687	.2050282	.1726574	.1227408
37	.2342969	.1961992	.1644356	.1157932
38	.2252854	.1877504	.1566054	.1092389
39	.2166206	.1796655	.1491479	.1030555
40	.2082890	.1719287	.1420457	.0972222
41	.2002779	.1645251	.1342816	.0917191
42	.1925749	.1574403	.1288396	.0865274
43	.1851682	.1506605	.1227044	.0816296
44	.1180464	.1441728	.1168613	.0770091
45	.1711984	.1379644	.1112965	.0726501
46	.1647139	.1320233	.1059967	.0685378
47	.1582826	.1263381	.1009492	.0646583
48	.1521948	.1208977	.0961421	.0609984
49	.1463411	.1156916	.9915639	.0575457
50	.1407126	.1107997	.08720371	.0542884

406. The-

406. Theorems *resolving all Questions relating to* Annuities, *&c.* *in* Arrear, *calculated at* Compound Interest.

Here $\begin{cases} R = \text{one Pound, and its Interest for one Year, as before.} \\ U = \text{the first Year's Rent without Interest.} \end{cases}$

Then $RU =$ the Amount of the first Year's Rent, and its Interest.

And hence is form'd the following Progression of Amounts in continued geometrical Proportion.

Thus $\begin{cases} 1. \quad 2. \quad 3. \quad 4. \quad 5. \quad \text{\&c. the Years.} \\ U + UR + UR^2 + UR^3 + UR^4, \ \text{\&c. the Amounts.} \end{cases}$

Hence $U + UR + UR^2 + UR^3 + UR = A$, the Amount of any yearly Rent, or Annuity, forborn five Years.

Now the last Term in the above Series is $UR^4 = UR^{t-1}$.

Therefore $A - UR^{t-1} =$ the Sum of all the Antecedents.

And $A - U =$ the Sum of all the Consequents in the Series.

So that it will be, $U : RU :: A - UR^{t-1} : A - U$.*

Therefore $AU - UU = RUA - UUR^t$. Divide all by U.

Then $A - U = RA - UR^t$. The General Theorem.

407. From hence we deduce the following particular Theorems, *viz.*

Given U, R, *t*, to find A?

Theorem 1. $\dfrac{UR_t - U}{R - 1} = A.$

Given A, R, *t*, to find U?

Theorem 2. $\dfrac{RA - A}{R^t - 1} = U.$

Given

* For in any geometrical Series, it will be as any one Antecedent is to its Consequent, so is the Sum of all the Antecedents to the Sum of all the Consequents; thus, let the Series be $a : ar :: ar : ar^2 :: ar^2 : ar^3 :: ar^3 : ar^4$, *&c*, Then as $a : ar :: a + ar + ar^2 + ar^3$, *&c.* $: ar + ar^2 + ar^3 + ar^4$, &c. for the Product of the two Extremes is evidently equal to the Product of the two Means. (138.)

Given U, A, R, to find *t*?

Theorem III. $\dfrac{RA + U - A}{U} = R^t$.

Given A, U, *t*, to find R?

Theorem IV. $\dfrac{A}{U} R - R^t = \dfrac{A - U}{U}$.

QUESTION I.

408. If 30*l*. yearly Rent be forborn, or unpaid nine Years, what will it amount to at the Rate of 6*l*. *per Cent.* &c. Compound-Intereſt?

Here is given $\left\{\begin{array}{l} U = 30 \\ t = 9 \\ R = 1{,}06 \end{array}\right\}$ To find A, *per* Theorem I.

In the firſt Place, let R =	- -	1,06	= 0,025305
Be involved to the 9th Power, (*viz.* R^t)	- -		9
That will be	- -	R^9 = 1,689451	= 0,227745
Multiply by	- - -	U = 30	= 1,477121
The Product is	- -	UR^t = 50,683530	= 1,704866
From that ſubtract	- -	U = 30	
The Remainder is the Dividend		= 20,68353	= $UR^t - U$.
Divide therefore	-	$UR^t - U$ = 20,68353	= 1,315626
By	- - -	R − 1 = 0,06	= 8,778151
The Quotient is	- -	A = *l*. 344,7267	= 2,537475

That is the Amount = 344*l*. 14*s*. 6¼*d*. the Anſwer.

QUESTION II.

409. What Annuity 3*l*. 10*s*. *per Cent.* Compound-Intereſt, will raiſe a Stock of 344*l*. 5*s*. being forborn eight Years?

Here is given $\left\{\begin{array}{l} A = 344{,}25 \\ R = 1{,}035 \\ t = 8 \end{array}\right\}$ To find U, *per* Theorem II.

Multiply the Amount - -	$A = 344,25$	$= 2,536874$
By the Rate - - - - -	$R = 1,035$	$= 0,014940$
From that Product -	$RA = 356,29875$	$= 2,551814$
Subtract the Amount -	$A = 344,25$	
The Remainder is -	$RA - A = 12,04875$, the Dividend.	
Then involve - - -	$R = 1,035$	$= 0,014940$
To the 8th Power, *viz.* R^t - - -		8
That Power will be - -	$R^8 = 1,316803$	$= 0,119520$
The same less Unity is	$R^t - 1 = 0,316803$, the Divisor.	
Therefore divide -	$RA - A = 12,04875$	$= 1,080908$
By - - - -	$R^t - 1 = 0,316803$	$= 9,500785$
The Quotient is - -	$U = 38,0297$	$= 1,580123$

The Annuity therefore which was sought, is found to be 38,0297*l.* = 38*l.* 0*s.* 7*d.* *per Ann.* Answer.

QUESTION III.

410. In what Time will 38*l.* 0*s.* 7*d.* raise a Stock of 344*l.* 5*s.* at 3*l.* 10*s.* *per Cent. per Ann.* Compound-Interest?

$$\text{Here is given}\left\{\begin{array}{l} U = 38,0297 \\ A = 344,25 \\ R = 1,035 \end{array}\right\}\text{To find } t, \textit{ per } \text{Theorem III.}$$

First multiply the Amount -	$A = 344,25$	$= 2,536874$
By the given Rate - -	$R = 1,035$	$= 0,014940$
To that Product -	$RA = 356,29875$	$= 2,551814$
Add the Annuity - -	$U = 38,0297$	
From the Sum -	$RA + U = 394,32845$	
Take the Amount -	$A = 344,25$	
The Remainder is the Dividend	$RA + U - A = 50,07845$	$= 1,699651$
Which divide by - -	$U = 38,0297$	$= 1,580123$
The Quotient will be -	$R^t = 1,316803$	$= 0,119528$

Then

Then divide 1,316803 continually by the Rate 1,035 until nothing remains, and the Number of thoſe Diviſions will be 8 $= t =$ Time required.

But much better by the Numbers in Table II. where under the given Rate 1,035, you will find the Number 1,316809, againſt which, by the Side, is eight Years, the Time required. (396, 397, 398.)

Question IV.

411. At what Rate *per Cent.* Compound-Intereſt, will 30*l.* yearly Rent, being forborn, or unpaid, nine Years, amount to 344*l.* 14*s.* 6¼*d.*

$$\text{Here is given} \left\{ \begin{array}{l} U = 30 \\ A = 344{,}7267 \\ t = 9 \end{array} \right\} \text{To find R, } per \text{ Theor. IV.}$$

Firſt divide the Amount -	$A = 344{,}7267$	$= 2{,}537475$
By the Annuity - -	$U = 30$	$= 1{,}477121$
The Quotient is - -	$\frac{A}{U} = 11{,}4909$	$= 1{,}000354$
Again the Amount leſs the Annuity is - -	$A - U = 314{,}7267$	$= 2{,}497933$
Which divide by - -	$U = 30$	$= 1{,}477121$
The Quotient is -	$\frac{A - U}{U} = 10{,}49014$	$= 1{,}020812$

Now $R^t = R^9$. Therefore the Theorem affords this Equation, *viz.* $11{,}4909\,R - R^9 = 10{,}49014$.

This being a very high Equation, requires the Aſſiſtance of Algebra, to be found farther on; but as this Value of R may be nearly determined by the Tables, with the utmoſt Eaſe, I ſhall next proceed to ſhew

The Construction of TABLE IV.

412. The Construction of Table IV. is from Theorem I. of Annuities, &c. in Arrears, *viz.* $\frac{UR^t - U}{R - 1} = A$. Now as it is $U = 1l.$ and $R = 1,05$, as before, then the Theorem will be brought to $\frac{R^t - 1}{,05} = A$, the Amount of 1*l.* Annuity for the Number of Years designed by *t*. That is, from (R^t) the Numbers in the second Table, subtract Unity, or 1. The Remainder, divided by ,05 (or $R - 1$) gives the Numbers in the fourth Table.

Example at 5 per Cent.

The Numbers in the 1st Table less Unity or 1, are	Which divided by ,05, the Quotients are	Amount of 1*l* Annuity for	Years, &c.
,05	1,00000	1	
,1025	2,05	2	
,157625	3,1525	3	
,21550625	4,310125	4	
,27628156	5,525631	5	

And thus you proceed for any other Rate of Interest.

413. The Use of this Table take in the following Instances.

Against 9 Years, and under 6 *per Cent.* you find the Number - - - - -	11,4913162
Which multiply by the Annuity - - - -	30
The Product is the present Worth -	£. 344,7394860

Which is nearly the same as *per* Theorem I, (408.)

414. Again; in the Equation $11,4909\,R - R^9 = 10,49014$, (411.) it will be easy to see, that R must be such a Rate in Table II. against 9 Years, that the Number answering to it, added to 10,49014, must be but little less than 11,4909; because $11,4909\,R = 10,49014 + R^9$: If we but slightly examine

examine the Numbers under 4, $4\frac{1}{2}$, and 5 *per Cent.* we find they are too ſmall; if we take that under 6 *per Cent.* *viz.* $R^9 = 1,689479$, and add to it 10,49014, the Sum is 12,17962, and $11,4909 \times 1,06 = 12,1803$, *&c.* nearly the ſame; and therefore ſhews the Rate allowed was 6 *per Cent.* as required.

This Inſtance is ſufficient to ſhew the extreme Uſe of Tables, not only to thoſe who do not underſtand Algebra, but even to thoſe who do.

TABLE IV.

Of Compound Interest.

415. *The* Amount *of one Pound* per Ann. *or* Annuity, *for* Years, *at the Rates of* 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, *and* 6, per Cent. per Annum.

Years.	*2 per Cent.*	*$2\frac{1}{2}$ per Cent.*	*3 per Cent.*	*$3\frac{1}{2}$ per Cent.*
1	1.0000000	1.0000000	1.0000000	1.0000000
2	2.0200000	2.0250000	2.0300000	2.0350000
3	3.0604000	3.0756230	3.0909000	3.1062250
4	4.1216080	4.1525156	4.1836270	4.2149429
5	5.2040402	5.2563285	5.3091358	5.3624659
6	6.3081210	6.3877367	6.4684099	6.5501522
7	7.4342834	7.5474302	7.6624622	7.7794075
8	8.5829691	8.7361159	8.8923360	9.0516866
9	9.7546284	9.9545188	10.1591061	10.3684958
10	10.9497210	11.2033818	11.4638793	11.7313931
11	12.1687154	12.4834663	12.8077957	13.1419919
12	13.4120897	13.7955530	14.1920296	14.6019616
13	14.6803315	15.1404418	15.6177904	16.1130303
14	15.9739381	16.5189528	17.0863242	17.6769864
15	17.2934169	17.9319267	18.5989139	19.2956809
16	18.6392853	19.3802248	20.1568813	20.9710297
17	20.0120719	20.8647304	21.7615877	22.7050158
18	21.4123124	22.3863487	23.4144354	24.4996913
19	22.8405586	23.9460074	25.1168684	26.3571805
20	24.2973698	25.5446576	26.8703745	28.2796818
21	25.7833172	27.1832740	28.6764857	30.2694707
22	27.2989835	28.8628559	30.5367803	32.3289022
23	28.8440632	30.5844273	32.4528837	34.4604137
24	30.4218625	32.3490379	34.4264702	36.6665282
25	32.0302997	34.1577639	36.4592643	38.9498567

TABLE IV.

The Amount of one Pound, Annuity, Compound-Interest.

Years.	4 *per C.*	4 ½ *per C.*	5 *per C.*	6 *per C.*
1	1.0000000	1.0000000	1.0000000	1.0000000
2	2.0400000	2.0450000	2.0500000	2.0600000
3	3.1216000	3.1370250	3.1525000	3.1836000
4	4 2464640	4.2781911	4.3101250	4.3746016
5	5.4163226	5.4707097	5.5256312	5.6370930
6	6.6329755	6.7168917	6.8019128	6.9753187
7	7.8982945	8.0191518	8.1420084	8.3938378
8	9.2142263	9.3800136	9.5491089	9.8974681
9	10.5827953	10.8021142	11.0265643	11.4913162
10	12.0061071	12.2882094	12.5778925	13.1807958
11	13.4863514	13.8411788	14.2067871	14.9716435
12	15.0258055	15.4640318	15.9171265	16.8699420
13	16.6268377	17.1599133	17.7129829	18.8821385
14	18.2919112	18.9321094	19.5986320	21.0150667
15	20.0235876	20.7840543	21.5785636	23.2759707
16	21.8245311	22.7143367	23.6574918	25.6725289
17	23.6975124	24.7417069	25.8403664	28.2128806
18	25.6454129	26.8550837	28.1323847	30.9056534
19	27.6712294	29.0635625	30.5390039	33.7599925
20	29.7780786	31.3714228	33.0659541	36.7855920
21	31.9692017	33.7831368	35.7192518	39.9927275
22	34.2479698	36.3033779	38.5052144	43.3922911
23	36.6178886	38.9370299	41.4304751	46.9958285
24	39.0826041	41.6891963	44.5019989	50.8155782
25	41.6459083	44.5652101	47.7270988	54.8645128

TABLE

TABLE IV.

The Amount of one Pound, Annuity, Compound-Interest.

Years.	2 per C.	2 ½ per C.	3 per C.	3 ½ per C.
26	33.6709057	36.0117080	38.5530422	41.3131017
27	35.3443238	37.9120007	40.7096335	43.7590602
28	37.0512103	39.8598008	42.93[illegible]9225	46.2906273
29	38.7922345	41.8562958	45.2188502	48.9107993
30	40.5680792	43.9027032	47.5754157	51.6226773
31	42.3794408	46.0002707	50.0026782	54.4294719
32	44.2270296	48.1502775	52.5027585	57.3345025
33	46.1115702	50.3540345	55.0778413	60.3412101
34	48.0338016	52.6128653	57.7301765	63.4531524
35	49.9944776	54.9282074	60.4620818	66.6740127
36	51.9943672	57.3014126	63.2759443	70.0076032
37	54.0342545	59.7339479	66.1742226	73.4578693
38	56.1149396	62.2272966	69.1594493	77.0288947
39	58.2372384	64.7829791	72.2342327	80.7249060
40	60.4019832	67.4025535	75.4012597	84.5502778
41	62.6100228	70.0876174	78.6632975	88.5095375
42	64.6822233	72.8398078	82.0231964	92.6073713
43	67.1594678	75.6608030	85.4838923	96.8486293
44	69.5026511	78.5523231	89.0484191	101.2383313
45	71.8927103	81.5161312	92.7198614	105.7816729
46	74.3305645	84.5540344	96.5014172	110.4840315
47	76.8171758	87.6678853	100.3965009	115.3549720
48	79.3535193	90.8595824	104.4083960	120.3882560
49	81.9405697	94.1310729	108.5406479	125.6018450
50	84.5794015	97.4843488	112.7968673	130.9979100

TABLE

TABLE IV.

The Amount of one Pound, Annuity, Compound-Interest.

Years.	4 *per C.*	4½ *per C.*	5 *per C.*	6 *per C.*
26	44.3117446	47.5706446	51.1134538	59.1563827
27	47.0842144	50.7113236	54.6691265	63.7057657
28	49.9675830	53.9933332	58.4025828	68.5281116
29	52.9662863	57.4230332	62.3227119	73.6397983
30	56.0849377	61.0070698	66.4388475	79.0581862
31	59.3283352	64.7523878	70.7607899	84.8016774
32	62.7014687	68.6662452	75.2988294	90.8897780
33	66.2095274	72.7562263	80.0637708	97.3431647
34	69.8579045	77.0322565	85.0669594	104.1837546
35	73.6522248	81.4966180	90.3203073	111.4347799
36	77.5983138	86.1639658	95.8363227	119.1208667
37	81.7022464	91.0413443	101.6281388	127.2681187
38	85.9703362	96.1382048	107.7095458	135.9042058
39	90.4091497	101.4644249	114.0950231	145.0584581
40	95.0255157	107.0303231	120.7997742	154.7619656
41	99.8265363	112.8466876	127.8397829	165.0476836
42	104.8195978	118.9247885	135.2317511	175.9505446
43	110.0123817	125.2764040	142.9933386	187.5075772
44	115.4128169	131.9138422	151.1430056	199.7580319
45	121.0293920	138.8499651	159.7001559	212.7435138
46	126.8705677	146.0982135	168.6851637	226.5081246
47	132.9453904	153.6726331	178.1194218	241.0986121
48	139.2632060	161.5879016	188.0253929	256.5645288
49	145.8337342	169.8593572	198.4266626	272.9584006
50	152.6670836	178.5030282	209.3479957	290.3359046

416. Theorems resolving all Questions relating to the *present Worth* of *Annuities*, *Pensions*, or *Leases*, to continue any Time, or in *Reversion* after a given Time.

The Business will be to get a Theorem which may contain P, the *present Worth*; for this Purpose, we must have Recourse to those Theorems which give the Value of A, the Amount, as formerly in *Simple Interest*. Now, in Article 389, we had $PR^t = A$; and in Article 407, we had $\frac{UR^t - U}{R - 1} = A$; therefore by equating these two Theorems, we get this general Theorem, *viz.*

$$PR^t = \frac{UR^t - U}{R - 1}$$

417. From whence we derive the following particular Theorems.

Given U, R, *t*, to find P?

$$\text{Theorem I.}\left\{\frac{U - \frac{U}{R^t}}{R - 1} = P.\right.$$

Given P, R, *t*, to find U?

$$\text{Theorem II.}\left\{\frac{\overline{PR^t \times R} - PR^t}{R^t - 1} = U.\right.$$

Given U, R, P, to find *t*?

$$\text{Theorem III.}\left\{\frac{U}{P + U - PR} = R^t.\right.$$

Given U, P, *t*, to find R?

$$\text{Theorem IV.}\left\{\frac{U}{P} = \frac{U}{P}R^t + R_t - R^t + 1.\right.$$

QUES-

QUESTION I.

418. What is 30*l.* yearly Rent, worth in ready Money, for its Continuance 7 Years, allowing 6*l. per Cent.* Compound Interest, to the Purchaser?

Here is given $\left\{\begin{array}{l} U = 30 \\ R = 1{,}06 \\ t = 7 \end{array}\right\}$ to find P, *per* Theorem I.

First, involve - - -	$R = 1{,}06$	$= 0{,}025305$
To the 7th Power, (*viz.* R^7) - - - -		7
That will be - -	$R^7 = 1{,}50061$	$= 0{,}177135$
Then divide - -	$U = 30$	$= 1{,}477121$
By R^t, there will remain -	$\frac{U}{R^t} = 19{,}9520$	$= 1{,}299986$
Then from the Annuity -	$U = 30$	
Subtract - - -	$\frac{U}{R^t} = 19{,}952$	
Remains the Dividend $U - \frac{U}{R^t}$	$= 10{,}048$	$= 1{,}002079$
Which divide by the Rate less Unity $= R - 1$	$= 0{,}06$	$= 8{,}778151$
The Quotient is the present Worth —	$P = 167{,}4716$	$= 2{,}223928$

The present Worth, in ready Money, is 167*l.* 9*s.* 5*d.* the Answer.

419. Suppose this were an Annuity in Reversion, or not to be entered on till after seven Years are past, and thence to continue seven Years; and you would know the present Worth; find by the second Theorem of Compound Interest (392.) what ready Money will amount to 167*l.* 9*s.* 5*d.* in seven Years, at the same Rate of Interest; and that will be its present Worth; and so for any other Annuity in Reversion.

The Construction of TABLE V.

420. It is but too easy to be observed, how very difficult the Solution of these very useful Questions must be in this common Way of Computation, and therefore we shall next shew, how a fifth Table may be constructed to facilitate this Affair. This is done from the general Theorem (416.)

$$PR^t = \frac{UR^t - U}{R - 1}$$

Whence we have $P = \frac{UR^t - U}{R^t \times R - 1} = \frac{R^t - 1}{0{,}05\ R^t}$ (by putting $U = 1l.$ and $R = 1{,}05$.) But $\frac{R^t - 1}{0{,}05}$ make the Numbers of Table IV. (412.) and R^t gives the Numbers of Table II. (396.) therefore the Numbers of Table V. arise from those of Table IV. divided by the Numbers in Table II. as in the following Examples at the Rate of 1,05 *per Cent.*

Table II.	Table IV.	Table V.
1,05)	1,00000	(= 0,95238, &c. for the 1st Year.
1,1025)	2,05000	(= 1,85941, &c. for the 2d Year.
1,157625)	3,15250	(= 2,72324, &c. for the 3d Year.
1,21550625)	4,310125	(= 3,54595, &c. for the 4th Year.
1,27628156)	5,525631	(= 4,32947, &c. for the 5th Year, &c.

And thus for any other Rate of Interest.

The

The Use of TABLE V.

421. What is the present Worth of 30*l.* *per Annum*, to continue seven Years, at 6*l.* *per Cent.*?

In the Table against 7 Years, and 6*l.* *per Cent.* you find - - - - - -	5,5823815
Which multiply by the Annuity 30*l.* - -	30
The Product is the present Worth - =	167,471445

Or 167*l.* 9*s.* 5*d.* as before. (418.)

This one Example is sufficient to shew the Use of this Table in any other Case.

TABLE V.

Of COMPOUND INTEREST.

422. *The* present Worth *of* 1l. per Ann. *or* Annuity, *for* Years, *at the Rates of* 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, *and* 6, per Cent. per Annum.

Years.	*2 per Cent.*	*$2\frac{1}{2}$ per Cent.*	*3 per Cent.*	*$3\frac{1}{2}$ per Cent.*
1	0.9803922	0.9756098	0.9708738	0.9661836
2	1.9415609	1.9274242	1.9134697	1.8996943
3	2.8838833	2.8560236	2.8286114	2.8016370
4	3.8077287	3.7619742	3.7170984	3.6730792
5	4.7134595	4.6458285	4.5797072	4.5150524
6	5.6014309	5.5081254	5.4171914	5.3285530
7	6.4719911	6.3493906	6.2302829	6.1145439
8	7.3254814	7.1701372	7.0196922	6.8739555
9	8.1622367	7.9708655	7.7861089	7.6076865
10	8.9825850	8.7520639	8.5302028	8.3166053
11	9.7868480	9.5142087	9.5256241	9.0015510
12	10.5753412	10.2577646	9.9540040	9.6633343
13	11.3483737	10.9831839	10.6349553	10.3027385
14	12.1062487	11.6909122	11.2960731	10.9205203
15	12.8492635	12.3813777	11.9379351	11.5174109
16	13.5777093	13.0550027	12.5611020	12.0941168
17	14.2918719	13.7121977	13.1661185	12.6513206
18	14.9920313	14.3533636	13.7535131	13.1896812
19	15.6784620	14.9788913	14.3237991	13.7098374
20	16.3514333	15.5891623	14.8774748	14.2124033
21	17.0112092	16.1845486	15.4150241	14.6979742
22	17.6580482	16.7654132	15.9369166	15.1671248
23	18.2922041	17.3321105	16.4436084	15.6204105
24	18.9139256	17.8849858	16.9355421	16.0583676
25	19.5234565	18.4243764	17.4131477	16.4815146

TABLE

TABLE V.

The present Worth of one Pound, Annuity, Compound-Interest.

Years.	4 *per* C.	$4\frac{1}{2}$ *per* C.	5 *per* C.	6 *per* C.
1	0.9615385	0.9569378	0.9523809	0.9433962
2	1.8860947	1.8726678	1.8594103	1.8333926
3	2.7750910	2.7489644	2.7232480	2.6730119
4	3.6298952	3.5875257	3.5459505	3.4651056
5	4.4518223	4.3899767	4.3294767	4.2123638
6	5.2421369	5.1578725	5.0756921	4.9173244
7	6.6020547	5.8927009	5.7863734	5.5823815
8	6.7327448	6.5958861	6.4632128	6.2097939
9	7.4353314	7.2687905	7.1078217	6.8016923
10	8.1108955	7.9127182	7.7217349	7.3600871
11	8.7604763	8.5289169	8.3064142	7.8868747
12	9.3850733	9.1185808	8.8632516	8.3838440
13	9.9856473	9.6828524	9.3925730	8.8526831
14	10.5631223	10.2228253	9.8986409	9.2949840
15	11.1183868	10.7395457	10.3796500	9.7122491
16	11.6522949	11.2340151	10.8377695	10.1058953
17	12.1656680	11.7071914	11.2740662	10.4772597
18	12.6592961	12.1599918	11.6895869	10.8276035
19	13.1339385	12.5932936	12.0853208	11.1581165
20	13.5903253	13.0079365	12.4622103	11.4699213
21	14.0091589	13.4047239	12.8211527	11.7640767
22	14.4511142	13.7844248	13.1630026	12.0415811
23	14.8568405	14.1477749	13.4885739	12.3033790
24	15.2469619	14.4954784	13.7986418	12.5503576
25	15.6220787	14.8282084	14.0939445	12.7833562

TABLE

TABLE V.

The present Worth of one Pound, Annuity, Compound-Interest.

Years	2 per C.	2 ½ per C.	3 per C.	3 ½ per C.
26	20.1210358	18.9506111	17.8768420	16.8903523
27	20.7068978	19.4640109	18.3270315	17.2853645
28	21.2812724	19.9648887	18.7641082	17.6670188
29	21.8443847	20.4535499	19.1884546	18.0357670
30	22.3964556	20.9302926	19.6004413	18.3920454
31	22.9377015	21.3954074	20.0004285	18.7362758
32	23.4683348	21.8491780	20.3887655	19.0688656
33	23.9885636	22.2918809	20.7657918	19.3902082
34	24.4985917	22.7237863	21.1318367	19.7006842
35	24.9986193	23.1451573	21.4872200	20.0006612
36	25.4888425	23.5562511	21.8322525	20.2904938
37	25.9694534	23.9573181	22.1672354	20.5705254
38	26.4406406	24.3486030	22.4924616	20.8410874
39	26.9025888	24.7303444	22.8082151	21.1024999
40	27.3554792	25.1027751	23.1147719	21.3550723
41	27.7994895	25.4661220	23.4123999	21.5991037
42	28.2347936	25.8206088	23.7013592	21.8348828
43	28.6615623	26.1664457	23.9819021	22.0626887
44	29.0799631	26.5038495	24.2542739	22.2827910
45	29.4901599	26.8330239	24.5187125	22.4954503
46	29.8923136	27.1541696	24.7754490	22.009181
47	30.2865820	27.4674826	25.0247078	22.994378
48	30.6731196	27.7731537	25.2667066	23.0912443
49	31.0520780	28.0713695	25.5016569	23.2765645
50	31.4236059	28.3623117	25.7297640	23.4556179

TABLE

TABLE V.

The present Worth of one Pound, Annuity, Compound-Interest.

Years.	4 per C.	$4\frac{1}{2}$ per C.	5 per C.	6 per C.
26	15.9827678	15.1466115	14.3751853	13.0031663
27	16.3295844	15.4513028	14.6430336	13.2105342
28	16.6630618	15.7428735	14.8981272	13.4061644
29	16.9837132	16.0218885	15.1410735	13.5907211
30	17.2920318	16.2888885	15.3724510	13.7648312
31	17.5884921	16.5443909	15.5928104	13.9290861
32	17.8735500	16.7888909	15.8026766	14.0840435
33	18.1476441	17.0228621	16.0025491	14.2302297
34	18.4111962	17.2467580	16.1929039	14.3681412
35	18.6646116	17.4610124	16.3741942	14.4982465
36	18.9082803	17.6660406	16.5468516	14.6209872
37	19.1425771	17.8622398	16.7112872	14.7367804
38	19.3678625	18.0499902	16.8678926	14.8460192
39	19.5844831	18.2296557	17.0170406	14.9490747
40	19.7927721	18.4015844	17.1590862	15.0462969
41	19.9930500	18.5661095	17.2943678	15.1380160
42	20.1856250	18.7235498	17.4232074	15.2245434
43	20.3707931	18.8742103	17.5459118	15.3061730
44	20.5488395	19.0183831	17.6627732	15.3831821
45	20.7200378	19.1563474	17.7740697	15.4558321
46	20.8846517	19.2883707	17.8800663	15.5243699
47	21.0429342	19.4147088	17.9610155	15.5890282
48	21.1951289	19.5356066	18.0771576	15.6500266
49	21.3414700	19.6512981	18.1687215	15.7075723
50	21.4821826	19.7620078	18.2559253	15.7618610

423. By Theorem II. (407.) a very uſeful Queſtion is reſolved, *viz.*

QUESTION II.

What Annuity, to continue ſeven Years, may be purchaſed for 120*l.* 5*s.* at 6 *per Cent.* Compound Intereſt.

Here is given $\left\{\begin{array}{l} P = 120{,}25 \\ R = 1{,}06 \\ t = 7 \end{array}\right\}$ to find U, the Annuity.

Involve the Rate - - $R = 1{,}06 = 0{,}025305$
To the Index of its Power (*viz.* $= t$) — 7

The Power of R will be $R^7 = 1{,}50361 = 0{,}177135$
Which multiply by the preſent Worth - - - . $P = 120{,}25 = 2{,}080084$

The Product is - - $P\,R^t = 180{,}8087 = 2{,}257219$
Multiply that by the Rate - - $R = 1{,}06 = 0{,}025305$

That Product is - $P\,R^t \times R = 191{,}65722 = 2{,}282524$
From which ſubtract - $P\,R^t = 180{,}8087$
There remains the Dividend - $10{,}84852 = P\,R^t \times R - P\,R^t$.

Divide therefore $P\,R^t \times R - P\,R^t = 10{,}84852 = 1{,}035369$
By the Power of R leſs $1 = R^t - 1 = 0{,}50361 = 9{,}702113$

The Quotient is the Annuity $U = 21{,}54057 = 1{,}333256$

The Annuity ſought therefore is 21,54057*l.* = 21*l.* 10*s.* 9½*d.*

424. But we ſhall next ſee, with how much greater Eaſe theſe Queſtions are ſolved by a proper Table; the Conſtruction of which is as follows.

The

The Construction of TABLE VI.

This Table is made from Theorem II. (407.) by putting $P = 1l.$ which then is reduced to this Form, $R R^t - R^t = U R^t - U$; whence (at five *per Cent.*) it will be $,05 R^t = U R^t - U$; consequently $\frac{,05 R^t}{R^t - 1} = U$, the Annuity required.

But this being just the Reverse of $\frac{R^t - 1}{,05 R^t}$, which make the Numbers of Table V. it is plain, these two Theorems, which constitute the Numbers of Table V. and VI. multiplied together can make but 1, that is $\frac{,05 R^t}{R^t - 1} \times \frac{R^t - 1}{,05 R^t} = 1$.

425. Hence then, if the Numbers of Table V. be made Divisors, and Unity, or 1, the constant Dividend, the Quotients shall be the Numbers which constitute the 6th Table, at 5 *per Cent.* and after the same Manner for any other Rate of Interest.

Examples at 5 per Cent.

The Numbers of Table V. at 5 *per C.* are	By which divide Unity, the Quotients will be	The Annuit. 1*l.* will purchase for	
,9523809	1,05	1	Years, &c.
1,8594103	,5378049	2	
2,7232480	,3637734	3	
3,5459505	,2787437	4	
4,3294767	,2277916	5	

426. The Use of the Table in solving the above Question, is shewn in three Lines only.

The tabular Number for the Time and Rate is	-	0,179135
Which multiplied by the proposed Sum	- -	120,25
Gives the Annuity to be purchased, *viz.*	- £.	21,54098

TABLE VI.

Of Compound Interest.

427. *The* Annuity *which* one Pound *will purchase for any Number of* Years, *at the Rates of* 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, *and* 6, per Cent. per Annum.

Years.	2 *per C.*	$2\frac{1}{2}$ *per C.*	3 *per C.*	$3\frac{1}{2}$ *per C.*
1	1.0200000	1.0250000	1.0300000	1.0350000
2	.5150495	.5188272	.5226108	.5264005
3	.3467547	.3501372	.3535304	.3569342
4	.2626238	.2658179	.2690271	.2722511
5	.2121584	.2152469	.2183546	.2214814
6	.1785258	.1815499	.1845975	.1876682
7	.1545120	.1574954	.1605064	.1635445
8	.1365098	.1394674	.1424564	.1454767
9	.1225154	.1254569	.1284339	.1314460
10	.1113265	.1142588	.1172305	.1202414
11	.1021779	.1051060	.1080775	.1110920
12	.0945596	.0974871	.1004621	.1034840
13	.0881183	.0910483	.0940295	.0970616
14	.0826020	.0855365	.0885263	.0915707
15	.0778255	.0807665	.0837666	.0868251
16	.0736501	.0765990	.0796109	.0826848
17	.0699698	.0729278	.0759525	.0790431
18	.0667021	.0696701	.0727087	.0758168
19	.0637818	.0667606	.0698139	.0729403
20	.0611567	.0641471	.0672157	.0703610
21	.0587847	.0617873	.0648718	.0680366
22	.0566314	.0596466	.0627474	.0659321
23	.0546681	.0576964	.0603139	.0640188
24	.0528511	.0559128	.0590474	.0622728
25	.0512204	.0542759	.0574279	.0606740

TABLE VI.

The Annuity which one Pound will purchase, Compound Interest.

Years.	4 *per Cent.*	4½ *perCent.*	5 *per Cent.*	6 *per Cent.*
1	1.0400000	1.0450000	1.0500000	1.0600000
2	.5301961	.5339976	.5378049	.5454369
3	.3603485	.3637734	.3672086	.3741098
4	.2754901	.2787437	.2820118	.2885915
5	.2246271	.2277916	.2309748	.2373964
6	.1907619	.1938784	.1970157	.2033626
7	.1666096	.1097015	.1728198	.1791350
8	.1485279	.1516097	.1547218	.1610359
9	.1344930	.1375745	.1406901	.1470222
10	.1232909	.1263788	.1295046	.1358680
11	.1141490	.1172482	.1203890	.1267929
12	.1065522	.1096662	.1128254	.1192770
13	.1001437	.1032754	.1064558	.1129601
14	.0946690	.0978203	.1010240	.1275849
15	.0899411	.0931138	.0963423	.1029628
16	.0858200	.0890154	.0922699	.0989521
17	.0821985	.0854176	.0386991	.0954448
18	.0789933	.0822369	.0855462	.0923565
19	.0761386	.0794073	.0827450	.0896209
20	.0735818	.0768761	.0802426	.0871846
21	.0712801	.0746006	.0779961	.0850046
22	.0691988	.0725457	.0759705	.0830456
23	.0673091	.0706825	.0741368	.0812785
24	.0655868	.0689870	.0724709	.0796790
25	.0640121	.0674390	.0709545	.0782267

TABLE

TABLE VI.

The Annuity which one Pound will purchase Compound-Interest.

Years.	2 *per Cent.*	$2\frac{1}{2}$ *perCent.*	3 *per Cent.*	$3\frac{1}{2}$ *perCent.*
26	0.0496992	0.0527688	0.0559383	0.0592054
27	0.0482931	0.0513769	0.0545642	0.0578524
28	0.0469897	0.0500879	0.0532932	0.0566027
29	0.0457784	0.0488913	0.0521147	0.0554454
30	0.0446499	0.0477776	0.0510193	0.0543713
31	0.0435964	0.0467390	0.0499989	0.0533724
32	0.0426106	0.0457683	0.0490466	0.0524415
33	0.0416865	0.0448594	0.0481561	0.0515724
34	0.0408187	0.0440068	0.0473220	0.0507597
35	0.0400022	0.0432056	0.0465393	0.0499984
36	0.0392329	0.0424516	0.0458038	0.0492842
37	0.0385068	0.0417409	0.0451116	0.0486133
38	0.0378206	0.0410701	0.0444593	0.0479821
39	0.0371711	0.0404362	0.0438439	0.047 878
40	0.0365558	0.0398362	0.0432624	0.0468273
41	0.0359719	0.0392679	0.0427124	0.0462982
42	0.0354173	0.0387288	0.0421917	0.0457983
43	0.0348899	0.0382169	0.0416981	0.0453254
44	0.0343879	0.0377304	0.0412299	0.0448777
45	0.0339096	0.0372675	0.0407852	0.0444534
46	0.0334534	0.0368268	0.0403625	0.0440511
47	0.0330179	0.0364067	0.0399605	0.0436692
48	0.0326018	0.0360060	0.0395778	0.0433065
49	0.0322040	0.0356235	0.0392131	0.0429617
50	0.0318032	0.0352581	0.0388655	0.0426337

TABLE

TABLE VI.

The Annuity which one Pound will purchase Compound-Interest.

Years	4 per C.	4 ½ per C	5 per C.	6 per C.
26	0.0625674	0.0660214	0.0695643	0.0769044
27	0.0612385	0.0647195	0.0682919	0.0756972
28	0.0600130	0.0635208	0.0671225	0.0745926
29	0.0588799	0.0624146	0.0660455	0.0735796
30	0.0578301	0.0613915	0.0650514	0.0726489
31	0.0568554	0.0604435	0.0641321	0.0717522
32	0.0559486	0.0595632	0.0632804	0.0710023
33	0.0551036	0.0587445	0.0624900	0.0702729
34	0.0543148	0.0579819	0.0617554	0.0695984
35	0.0535773	0.0572705	0.0610717	0.0689739
36	0.0528869	0.0566058	0.0604345	0.0683948
37	0.0522396	0.0559840	0.0598398	0.0678574
38	0.0516319	0.0554017	0.0592842	0.0673581
39	0.0510608	0.0548557	0.0587646	0.0668938
40	0.0505235	0.0543431	0.0582782	0.0664915
41	0.0500174	0.0538616	0.0578223	0.0660589
42	0.0495402	0.0534087	0.0573947	0.0656834
43	0.0490899	0.0529824	0.0569933	0.0653331
44	0.0486645	0.0525807	0.0566163	0.0650061
45	0.0482625	0.0522020	0.0562617	0.0647005
46	0.0478821	0.0518447	0.0559282	0.0644149
47	0.0475219	0.0515073	0.0556142	0.0641477
48	0.0471807	0.0511886	0.0553184	0.0638977
49	0.0468571	0.0508872	0.0550397	0.0636636
50	0.0465502	0.0506021	0.0547767	0.0634443

428. By

428. By Theorem III. (417.) you find the Time for which a proposed Annuity may be purchased, for a given Sum, at a given Rate of Interest, as in the following Examples.

QUESTION III.

For what Time will 167*l.* 9*s.* 5*d.* purchase an Annuity of 30*l.* *per Ann.* at 6 *per Cent.* Compound Interest?

$$\text{Here is given}\left\{\begin{array}{l}P = 167{,}4716 \\ U = 30 \\ R = 1{,}06\end{array}\right\}\text{to find } t, \textit{ per } \text{Theor. III.}$$

To the present Worth - -	$P =$	$167{,}4716$	
Add the Annuity - -	$U =$	30	
The Sum is - -	$P + U =$	$197{,}4716$	
Then multiply the present Worth	$P =$	$167{,}4716 =$	$2{,}223928$
By the Rate - - - -	$R =$	$1{,}06 =$	$0{,}025305$
The Product is - -	$PR =$	$177{,}5199 =$	$2{,}249233$
Which subtract from Sum of	$P + U =$	$197{,}4716$	
The Remainder is	$P + U - PR =$	$19{,}9517$, the Divisor.	
Then divide the Annuity - -	$U =$	$30 =$	$1{,}477121$
By the Divisor -	$P + U - PR =$	$19{,}9517 =$	$1{,}299986$
The Quotient is - -	$R^t =$	$1{,}50361 =$	$0{,}177135$

Then in Table II. under 6 *per Cent.* you find the Number 1,5036303, against which, in the Side is 7 Years, the Answer.

QUESTION IV.

429. Suppose I purchase an Annuity of 21*l.* 10*s.* 9½*d.* to continue 7 Years, for 120*l.* 5*s.* ready Money; at what Rate *per Cent.* Compound Interest, was the Purchase made?

$$\text{Here is given}\left\{\begin{array}{l}P = 120{,}25 \\ U = 21{,}54057 \\ t = 7\end{array}\right\}\text{to find R, } \textit{per } \text{Theor. IV.}$$

First,

First, divide the Annuity - $U = 21,54057 = 1,333256$
By the present Worth - $P = 120,25 = 2,080084$

The Quotient is - - $\frac{U}{P} = 0,17915 = 9,253172$

Then multiply it into the given Power of the Rate, to which add the Power, *&c.* as *per* Theorem; and you have this Equation, *viz.* $0,17915\, R^7 + R^7 - R^8 = 0,17915$; or $1,17915\, R^7 - R^8 = 0,17915$; whence R will be found (by Table II.) after the same Manner as in (414.) to be 1,06, or that the Rate is 6 *per Cent.*

CHAP. XX.

THEOREMS *resolving all Questions relating to the purchasing of* Freehold, *or* Real Estates, *at* Compound-Interest.

430. TO purchase a Freehold Estate is evidently nothing more than to find the present Worth of an Annuity, to continue for ever; and consequently the Theor. (416.) $P R^t = \frac{U R^t - U}{R - 1}$, will also serve our Purpose here, if we make the Index (t) infinite (as in this Case the Time really is;) for then (since a finite Quantity, U, subtracted from an infinite one, makes no Alteration) we have

$$P R^t = \frac{U R^t}{R - 1}; \text{ or, } \overline{PR - P} \times R^t = U R^t.$$

That is, $PR - P = U$, Theorem I.

And $\frac{U}{R - 1} = P$, Theorem II.

 And

And $\frac{P + U}{P} = R$, Theorem III.

By these Theorems, the following useful Questions are solved.

QUESTION I.

431. Suppose a Freehold Estate of 25*l.* *per Ann.* were to be sold, what is the Worth, allowing 5*l.* 10*s.* *per Cent.* &c. Compound-Interest to the Buyer?

Here is given $\left\{ \begin{array}{l} U = 25 \\ R = 1{,}055 \end{array} \right\}$ To find P, *per* Theorem II.

Divide the annual Rent - -	$U = 25$	$= 1{,}397940$
By the Rate less Unity -	$R - 1 = 0{,}055$	$= 8{,}740362$
The Quotient is the Worth -	$P = 454{,}5$	$= 2{,}657578$

The Value of that Estate therefore is 454*l.* 10*s.* 10¾*d.* Q. E. I.

QUESTION II.

432. Suppose a Person would lay out 41ɸ*l.* 13*s.* 4*d.* on a Freehold Estate, and so as to be allowed 6*l.* *per Cent.* for his Money, Compound-Interest; what must be the annual Rent of such an Estate?

Here is given $\left\{ \begin{array}{l} P = 41ɸ{,}6 \\ R = 1{,}06 \end{array} \right\}$ To find U, *per* Theorem I.

Multiply the present Worth -	$P = 41ɸ{,}6$	$= 2{,}619789$
By the Rate - -	$R = 1{,}06$	$= 0{,}025305$
The Product is - -	$PR = 441{,}ɸ$	$= 2{,}645094$
From which subtract the Worth	$P = 41ɸ{,}6$	

There remains the annual Rent. The Answer. $\}$ U = 25*l.* *per Ann.*

QUES

QUESTION III.

433. Suppose one give 416*l.* 13*s.* 4*d.* for a Freehold Estate of 25*l.* *per Ann.* what Rate *per Cent.* Compound-Interest, has the Purchaser for his Money?

Here is given $\left\{\begin{array}{l} P = 41\not{6},6 \\ U = 25 \end{array}\right\}$ To find R, *per* Theor. III.

To the present Worth	-	$P = 41\not{6},6$
Add the annual Rent	-	$U = 25,0$
Divide their Sum	-	$P + U = 441,\not{6} = 2,645094$
By the present Worth	-	$P = 41\not{6},6 = 2,619789$
The Quotient is the Rate sought		$R = 1,06 = 0,025305$

Then say, As 1*l.* : ,06*l.* :: 100*l.* : 6*l.* *per Cent.* the Answer.

434. That nothing may be wanting to facilitate the Computations of Simple and Compound-Interest, I shall here subjoin two Tables, to express the Time in Parts of a Year, without any Trouble. One of which will shew the Number of Days from the Beginning of the Year to any Day of a Month proposed; and the other will shew what Decimal Part of a Year they make. The first here follows, with its Use.

435. *A* TABLE

435. *A* TABLE *of* DAYS *for any given Time less than a Year.*

Days.	January	February	March	April	May	June	July	August	September	October	November	December
1	1	32	60	91	121	152	182	213	244	274	305	335
2	2	33	61	92	122	153	183	214	245	275	306	336
3	3	34	62	93	123	154	184	215	246	276	307	337
4	4	35	63	94	124	155	185	216	247	277	308	338
5	5	36	64	95	125	156	186	217	248	278	309	339
6	6	37	65	96	126	157	187	218	249	279	310	340
7	7	38	66	97	127	158	188	219	250	280	311	341
8	8	39	67	98	128	159	189	220	251	281	312	342
9	9	40	68	99	129	160	190	221	252	282	313	343
10	10	41	69	100	130	161	191	222	253	283	314	344
11	11	42	70	101	131	162	192	223	254	284	315	345
12	12	43	71	102	132	163	193	224	255	285	316	346
13	13	44	72	103	133	164	194	225	256	286	317	347
14	14	45	73	104	134	165	195	226	257	287	318	348
15	15	46	74	105	135	166	196	227	258	288	319	349
16	16	47	75	106	136	167	197	228	259	289	320	350
17	17	48	76	107	137	168	198	229	260	290	321	451
18	18	49	77	108	138	169	199	230	261	291	322	352
19	19	50	78	109	139	170	200	231	262	292	823	353
20	20	51	79	110	140	171	201	232	263	293	324	354
21	21	52	80	111	141	172	202	233	264	294	325	355
22	22	53	81	112	142	173	203	234	265	295	326	356
23	23	54	82	113	143	174	204	235	266	296	327	357
24	24	55	83	114	144	175	205	236	267	297	328	358
25	25	56	84	115	145	176	206	237	268	298	329	359
26	26	57	85	116	146	177	207	238	269	299	330	360
27	27	58	86	117	147	178	208	239	270	300	331	361
28	28	59	87	118	148	179	209	240	271	301	332	362
29	29		88	119	149	180	210	241	272	302	333	363
30	30		89	120	150	181	211	242	273	303	334	364
31	31		90		151		212	243		304		365

The

The USE *of the* TABLE.

436. First; To know the Number of Days from the Beginning of the Year, to any given Day of any Month.

This is obtained by Inspection only; thus from *January* the 1st, to *September* the 7th, is 250 Days; to *November* the 27th are 331, *&c.*

Secondly, To know what is the Number of Days from any given Day of any Month, to the End of the Year.

Suppose *September* 7, then from - -	365
Subtract the Number answering to *September* 7	250
There remains the Number of Days sought, *viz.*	115 Days.

Thirdly, To find the Number of Days between the given Day of any one Month, and any given Day of any other Month, in the same Year.

For Instance, To know how many Days there are between *April* the 17th and *October* 23.

Thus, from the Number answering to *October* 23 -	296
Subtract that answering to *April* 17. - -	107
The Remainder is the Number of Days sought.	189

Fourthly, To find the Number of Days from any given Day of any Month in one Year, to any given Day of any Month in the next Year.

How many Days is it from *September* the 7th in one Year, to *April* the 19th in the next?

From

From the Days of a whole Year - -	365
Subtract the Number to *September* 7 -	250
Remains the Number to the End of the Year	115
To which add the Number to *April* 19 -	109
The Sum is the Number of Days required.	224

And thus is the Number of Days readily found for any Interval of Time given, in the same Year compleatly; or which is Part of one, or Part of another Year.

437. Having then the Number of Days, it is easy to find what Decimal Part of the Year they make, in the following Table; and having found that, you have the Symbols, T, *t*, in the foregoing Theorems, representing any Part of a Year.

TABLE

438. *A* DECIMAL TABLE *of Days and Months in a Year.*

		10 Days	20 Days	30 Days	40 Days	50 Days	60 Days	70 Days	80 Days	90 Days	
0		.027397	.054795	.082192	.109589	.136986	.164384	.191781	.219178	.246575	0
1	.002740	.030137	.057534	.084932	.112329	.139726	.167123	.194521	.221918	.249315	1
2	.005479	.032877	.060274	.087671	.115068	.142466	.169863	.197260	.224658	.252055	2
3	.008219	.035616	.063014	.090411	.117808	.145205	.172603	.2	.227397	.254795	3
4	.010959	.038356	.065753	.093151	.120548	.147945	.175342	.202740	.230137	.257534	4
5	.013699	.041096	.068493	.095890	.123288	.150685	.178082	.205479	.232877	.260274	5
6	.016438	.043836	.071233	.098630	.126027	.153425	.180822	.208219	.235616	.263014	6
7	.019178	.046575	.073973	.101370	.128767	.156164	.183562	.210959	.238356	.265753	7
8	.021918	.049315	.076712	.104110	.131507	.158904	.186301	.213699	.241096	.268493	8
9	.024658	.052055	.079452	.106849	.134247	.161644	.189041	.216438	.243836	.271233	9

Months.	Decim.	Months.	Decim.	Months.	Decim.	Months.	Decim.		
1	.083333	4	.333333	7	.583333	10	.833333	100	.273973
2	.166667	5	.416667	8	.666667	11	.916667	200	.547945
3	.25	6	.5	9	.75	12	1.	300	.821918
								365	1.

The Use of the foregoing Table.

439. The decimal Part of a Year for Days under 10, is contained in the firſt Column; thus, for ſeven Days it is 0.019178.

For any Number of Days from 10 to 100, look for the even Tens on the Top, and the Units on the Side of the Table; and where they meet in the Table is the Number expreſſing the decimal Part of the Year; thus the Decimal of 47 Days is found under 40 on the Top and againſt ſeven on the Side, to be 0,128767. The Decimal of 73 Days is 0.2.

In the lower Part of the Table are the Decimals of a Year for Months; and 100, 200 or 300 Days. So that for three Calendar Months and 27 Days, the Decimal of a Year is $0{,}25 + 0.073973 = 0.323973$. Or for 275 Days, it is $0.547945 + 0.205479 = 0.7503424$.

440. As it is impoſſible to avoid the Uſe of Decimals in *Money* as well as in *Time*, in the Computations of Intereſt, as appears in all the preceeding Caſes and Examples; therefore it is expedient to ſubjoin the following Table of the dicimal Parts of a Pound Sterling, anſwering to any Number of Shillings, Pence, and Fathings, propoſed.

Thus, ſuppoſe it is required to expreſs 57 *l.* 17 *s.* 9 *d.* $\frac{3}{4}$ in Decimals. You take from the Table the correſponding Decimals for the reſpective Parts, and then add them together, as below.

		£.
The integral Part		57.
againſt	17 *s.* is	0.85
	9 *d.* $\frac{3}{4}$	0.040625
The Sum in Decimals		57.890625

441. Again, by the ſame Table any given decimal Sum may be reſolved into the common Species of Money; as the Sum 67,185416.

The

		£.
The integral Part is		67.
From the Decimal		0,185416̸
Take the next leſs for 3 *s.*	=	0,15
The Remainder anſwers to 8 ½	=	0,035416̸

Therefore the Sum £. 67,185416̸ = 67 *l.* 3 *s.* 8 *d.* ½.

442. *A Table of the Decimal Parts of a Pound Sterling.*

A DECIMAL TABLE
Of Money; *one* Pound *the Integer.*

S.	*Decimals.*	*P. q.*	*Decimals.*	*P. q.*	*Decimals.*
1	,05	0 $\frac{1}{8}$	,0005208	6,0	,025
2	,1	0 $\frac{1}{4}$	,0010416̸	6 $\frac{1}{4}$	,0260416̸
3	,15	0 $\frac{1}{2}$	,00208ʒ	6 $\frac{1}{2}$	,027008ʒ
4	,2	0 $\frac{3}{4}$	,003125	6 $\frac{3}{4}$	,028125
5	,25	1,0	,00416̸	7,0	,0291 6̸6
6	,3	1 $\frac{1}{4}$	,0052	7 $\frac{1}{4}$	,030208ʒ
7	,35	1 $\frac{1}{2}$	,00625	7 $\frac{1}{2}$	,03125
8	,4	1 $\frac{3}{4}$	,00729	7 $\frac{3}{4}$	,0322916̸
9	,45	2,0	,008ʒ3	8,0	,0ʒ3333
10	,5	2 $\frac{1}{4}$	,009375	8 $\frac{1}{4}$	,034375
11	,55	2 $\frac{1}{2}$	,010416̸	8 $\frac{1}{2}$	,035416̸
12	,6	2 $\frac{3}{4}$	,011458ʒ	8 $\frac{3}{4}$	,036458ʒ
13	,65	3,0	,0125	9,0	,0375
14	,7	3 $\frac{1}{4}$	,0135416̸	9 $\frac{1}{4}$	,0385416̸
15	,75	3 $\frac{1}{2}$	,01458ʒ	9 $\frac{1}{2}$	,03958ʒ
16	,8	3 $\frac{3}{4}$	,015625	9 $\frac{3}{4}$	,040625
17	,85	4,0	,016̸666	10,0	,0416̸66
18	,9	4 $\frac{1}{4}$	,017708ʒ	10 $\frac{1}{4}$	,042708ʒ
19	,95	4 $\frac{1}{2}$	,01875	10 $\frac{1}{2}$	,04375
		4 $\frac{3}{4}$	,0197916̸	10 $\frac{3}{4}$	,0447916̸
		5,0	,0208ʒ	11,0	,0458ʒ3
		5 $\frac{1}{4}$	,021875	11 $\frac{1}{4}$	,046875
		5 $\frac{1}{2}$	,022916̸	11 $\frac{1}{2}$	,047916̸
		5 $\frac{3}{4}$	,023958ʒ	11 $\frac{3}{4}$	,048958ʒ

443. An Example in Simple and Compound Interest, will make the whole Matter easy and conspicuous.

EXAMPLE I.

What will 65*l.* amount to, being lent from *March* the 7th, to *November* the 3d, at 5*l.* *per Cent. per Annum*, Simple-Interest?

From *March* the 7th to *November* the 3d are 241 Days, by (435); those make 8 Months, 2 Weeks, and three Days, = 0,660273 Decimals of a Year, by (438.) Then by Theorem I. (270)

Multiply the Time - - -	$T =$	0,660273
By the *Ratio* of the Rate - -	$R =$	0,05
And that Product - - -	$TR =$	0,03301365
Multiply by the Principal - -	$P =$	65
The Product is - - -	$TRP =$	2,14588725
To which add the Principal - -	$P =$	65
The Sum is the Amount sought -	$=$	67,1458*l.* &c.

EXAMPLE II.

What is the Amount thereof at Compound-Interest, the Rate and Time being the same?

The Logarithm of the Rate -	$R = 1{,}05 =$	0,0211893
Multiply by the Time -	$t =$	,6603
The Product is the Logar. of	$R^t = R^{0{,}6603} =$	0,0139912
To which add the Log. of the Prin.	$P = 63 =$	1,8129133
The Sum is the Log. of Amount	$A = 67{,}1281 =$	1,8269045

And thus the Theorems serve to answer Questions, when the Time is only Part of a Year, as well as when complete Years.

444. I shall now add a few Questions of a *mix'd Nature*, and which frequently happen, in order to shew the more extensive Use of the Tables.

QUES-

QUESTION I.

Suppose I have 790*l.* to be paid me within seven Years, in this Manner; at the End of the first Year 90*l.* of two Years 100*l.* of four Years 200*l.* and of seven Years 400*l.* Query, what the present Worth of those several Payments is in ready Money, allowing $4\frac{1}{2}$ *per Cent.* Compound Interest?

In Table III. the present Worth of 1*l.* at $4\frac{1}{2}$ *per Cent.* due at the End of one Year, is - -		0.95693778
Which multiply by the Principal - - - -		90
The Product is the present Worth of 90*l.*	=	86.124402
Thus the present Worth of 100*l.* due at the End of two Years, is found - -	=	91.57299
Also, of 200*l.* at the End of four Years	=	167.71226
And of 400*l.* at the End of seven Years	=	293.93140
The Sum of all these is - - - -	*l.*	639.341052

Which answers the Question, *viz.* 639*l.* 6*s.* $9\frac{3}{4}$*d.*

QUESTION II.

445. *A* owes to *B* 455*l.* to be paid in 14 Years, *viz.* at the End of every two Years 65*l.* But he would agree to pay him in seven Years, by *equal Payments* each Year; which *B* agrees to, and at the Rate of 6 *per Cent. Compound Interest.* Query, what the *annual Payment* must be?

1. Find the *present Worth* (by Table III.) of the seven Payments which were at first to be made, as *per Quest.* I. which you will find to be 293*l.* 5*s.* 2*d.*
2. Then find (by Table VI.) what *Annuity*, to continue seven Years at the given Rate, 293*l.* 5*s.* 2*d.* will purchase; which you will find to be 52*l.* 10*s.* 8*d.* and is the Answer to the Question.

QUESTION III.

446. *A* has a Term of seven Years in an Estate of 35*l.* *per Annum.* *B* has a Term of 14 Years in the same Estate in *Reversion* after the seven Years; and *C* has a farther Term of

20

20 Years in *Reversion* after the 21 Years. Query the present Values of the *several Terms*, at the Rate of 5 *per Cent. per Annum*?

By *Table* V. the *present Value* of 35 *l. per Annum*, may be found,

	l.	*s.*	*d.*
for 41 Years, to be	605	6	$0\frac{1}{2}$
for 21 Years, to be	448	14	$9\frac{1}{2}$
for 7 Years, to be	202	10	$5\frac{1}{2}$

Which subtract from each other, and it will appear,

	l.	*s.*	*d.*
That the *present Value* of *A*'s Term is	202	10	$5\frac{1}{2}$
of *B*'s Term	246	4	4
of *C*'s Term	156	11	3
For these *Values* answer the Question *l.*	605	6	$0\frac{1}{2}$

QUESTION IV.

447. Which is most advantagious a *Term* of 15 Years in an Estate of 100 *l. per Ann.* or the *Reversion* of such an Estate for ever after the Expiration of the said 15 Years; computing at the Rate of 5 *per Cent. per Ann.* Compound Interest?

	£.
An Estate of 100*l. per Ann.* in Fee Simple, at 5 *per Cent.* is worth - - -	2000
In Table V. the present Value of the same Estate at the same Rate, for 15 Years, is	1037,9658
The Difference is	962,0342

Now this Difference being the Value of the Reversion, it appears that the first Term of 15 Years is better than the Reversion for ever afterwards by 75,9316*l.* = 75*l.* 18*s.* $7\frac{1}{2}$*d.* Answer.

QUESTION V.

448. A Person having 12 Years to come in a Lease of an Estate of 60*l. per Ann.* for 40 Years, would know what present Money he must pay in order to renew or complete the Lease, by adding

adding 28 Years thereto, computing at 6 *per Cent.* Compound Interest?

	£.
By Table V. the present Value of 1*l.* *per Ann.* at 6 *per Cent.* for 40 Years, is	15.046297
By the same Table the Value of 1*l.* *per Ann.* at that Rate, for 12 Years to come, is	8.383844
The Difference is	6.662453
Which multiplied by	60
The Product is the Answer, *viz.*	399.747180

In Money, 399*l.* 14*s.* 11*d.*

QUESTION VI.

449. *A.* gives 1550*l.* for an Annuity of 100*l.* *per Annum* for 50 Years. *B.* puts 1550*l.* out at Interest. It is required to know which will amount to the greatest Sum at the End of the 50 Years, at the Rate of 6*l.* *per Cent.* &c. Compound Interest?

	£.
By Table IV. the Amount of 100*l.* Annuity, in 50 Years at 6 *per Cent.* may be found to be	29033,59046
By Table II. it may be found, that the Amount of 1550*l.* for that Time and Rate will be	28551,23885
Hence *A's Annuity* is more than *B*'s 1550*l.* by	482,35161

at the End of 50 Years. The present Value of which Difference is found, by Table III. to be 26*l.* 3*s.* 8½*d.* and so much was *A*'s Case *better* than *B*'s.

QUESTION VII.

450. What Annuity, to continue 14 Years, may be purchased with 1000*l.* due at the End of five Years; the Annuity to commence presently, at 5*l.* *per Cent.*?

		£.
By Table III. the present Worth of 1000 *l.* due five Years hence at 5 *per Cent.* may be found	=	783,5262
By Table VI. it may be found, that the Annuity which 783,5262 *l.* will purchase for 14 Years, at the Rate of 5 *per Cent.* is ——	=	79,1518

In Money, 79 *l.* 3 *s.* $0\frac{1}{4}$ *d.* *per Annum*, the Answer.

QUESTION VIII.

451. For a Lease of certain Profits for seven Years, *A*, makes *two* Offers, either to pay 150 *l.* as a *Fine*, and 300 *l.* *per Annum*; or 1700 *l.* *Fine*, without any *Rent*. *B*, bids 650 *l.* *Fine*, and 200 *l.* *per Annum*. And *C*, offers 200 *l.* *Fine*, and 405 *l.* *per Annum*. Query which is the *best* Offer, and what the *Difference*, computing at 5 *l.* *per Cent.* &c. *Compound Interest?*

	£.
1. By Table II. the Amount of 150 *l.* in seven Years, at 5 *per Cent.* may be found to be	211,0659
By Table IV. the Amount of 300 *l.* *per Ann.* in seven Years at the given Rate may be found	242,6025
Therefore *A*'s Offer, at the End of seven Years would be —— ——	2453,6684
2. By Table II. the Amount of 1700 *l.* in seven Years. (*A*'s second Offer) at the said Rate, is found to be - - - - -	2392,0802
3. By Table II. the Amount of 650 *l.* in seven Years, at the given Rate, will be found to be	914,6189
By Table IV. the Amount of 200 *l.* *per Ann.* in 7 Years, at that Rate, will be found to be	1628,4016
Therefore *B*'s Offer will, in 7 Years, amount to	2543,0205
4. By Table II. the Amount of 200 *l.* in seven Years, at the given Rate, will be found to be	281,4212
By Table IV. the Amount of 405 *l.* *per Ann.* for the given Time and Rate, will be found to be	3297,5132
So that *C*'s Offer, in 7 Years, will amount to	3578,9344

The

The Amounts therefore of the ſaid Offers, at the End of the ſaid Term, being thus known, the preſent Worth of the ſeveral Amounts, may be found by Table III. which are as follow.

		l.	*s.*	*d.*
The preſent Worth of	*A*'s firſt Offer will be	1885	18	3
	A's ſecond Offer	1700	00	0
	B's Offer -	1807	5	6
	C's Offer - - -	2543	9	8

Therefore the preſent Worth of what *C.* offers is more than

	l.	*s.*	*d.*
A's firſt Offer, by	657	11	5
A's ſecond Offer, by	843	9	8
B's Offer, by -	736	4	2

Which fully anſwers the Queſtion.

N. B. This Queſtion might be more readily anſwered by finding the preſent Worths of the ſeveral offered Annuities (as *per* Table V.) and adding to them the ſeveral Fines, as the Reader may try at his Leiſure.

QUESTION IX.

452. What Annuity is ſufficient to pay off a Debt of 50 Millions in 30 Years, at 4*l.* *per Cent.* Compound Intereſt?

In Table IV. againſt 30 Years, under 4 *per C.* is 0,0578301
Which multiply by the Debt - - - - 50000000

The Product is the Annuity ſought, *viz.* *l.* 2891505 *per Annum.*

So that ſuppoſing the National Debt to be 50 Millions, and the Intereſt paid to be two Millions *per Ann.* or 4*l.* *per Cent.* then will a Sinking Fund of 891505*l.* *per Ann.* clear the whole Debt in 30 Years.

N. B. By this Example appears the Neceſſity of continuing the tabular Numbers to ſo many Places of Decimals.

QUESTION X.

453. Suppose one Farthing had been lent at Compound Interest at 5 *per Cent.* in the first Year of the Christian Æra, or Birth of Christ, and so continued to the Year 1750; Query, the Amount thereof?

N. B. Having said enough about the Use of the Tables, I here intend only to give the Reader a Hint of the surprising Nature of Numbers in geometrical Proportion. (See Theor. I. Art. 390.)

		l.
Therefore, the Logarithm of the Rate	1,05 =	0,0211893
Multiplied by the Time	- - - - -	1750
The Product is	- - - - -	37,0812750
To which add the Logarithm of one Farthing, or the ,0010416 Part of a Pound, *viz.* - - - -	=	,7,0177288
The Sum is the Log. of the Amount sought	=	34,0990031

Now the Index of this Logarithm being 34, shews the Number of Figures, of which the Amount of one Farthing in the given Time doth consist, to be 35, of which let it be sufficient to express the five first in Figures; the rest in Cyphers; then will the said Amount be

12561000000000000000000000000000000*l.*

Now the Value of a solid Body, perfectly spherical, whose Diameter is 8000 *English* Miles, (which is somewhat bigger than the Diameter of the Globe of our Earth.) I say such a solid Body of fine Gold would be in Value about

2386600000000000000000000000*l.*

Now if from each of these great Numbers be cut off 23 Cyphers, the remaining Figures will be 125610000000 in the Amount of the Farthing; and 23866 in the Value of the Globe of Gold. But 23866) 125610000000 (= 5260000 nearly.

Hence

Hence it appears, that one ſingle Farthing put out to Uſe in the Manner aforeſaid, would amount to more in Value than five Millions and two hundred and ſixty thouſand Globes of fine ſolid Gold, each bigger than the Globe of the Earth! A ſtrange and ſurpriſing, but no leſs certain Truth! And this immenſe Amount would be greatly increaſed by inlarging the Rate of Intereſt.

454. As it may be ſome Curioſity to the Reader to know when theſe wonderful Queſtions firſt came in Uſe, and on what Occaſion, I ſhall conclude this Subject with an Account thereof from the learned Dr. *Wallis*, and in his own Words, which he ſubjoins to his Solution of the Queſtion of *buying a Horſe by giving a Farthing fer the firſt Nail, and doubling the Sum for every Nail in his Four Shoes.* His Words are:

" The firſt Occaſion of which Queſtion, I believe to be what I have cited, *Cap.* 13. of my *Opus Arithmeticum*, from *Alſephad* (an *Arabic* Writer) in his *Commentaries* upon *Tograius's* Verſes: Namely, That one *Seſſa*, an *Indian*, having firſt found out the Game at *Cheſſe*, and ſhewed it to his Prince *Shebram:* The King, who was highly pleaſed with it, bid him aſk what he would for the Reward of his Invention; whereupon he aſked, that for the firſt little Square of the Cheſſe-board, he might have one Grain of Wheat given him; for the ſecond, two; and ſo on doubling continually, according to the Number of Squares in the Cheſſe-board, which was 64. And when the King, who intended to give a very noble Reward, was much diſpleaſed, that he had aſked ſo trifling a one; *Seſſa* declared, that he would be contented with this ſmall one. So the Reward he had fixed upon, was ordered to be given him: But the King was quickly aſtoniſhed, when he found that this would riſe to ſo vaſt a Quantity, that the whole Earth itſelf could not furniſh out ſo much Wheat. But how great the Number of theſe Grains is, may be found by doubling one continually 63 Times, ſo that we may get the Number that comes in the laſt Place; and then one Time more yet to have the Sum of all. For the Double of the laſt Term (leſs by one) is the Sum of all. Now this will be more expeditiouſly done by Logarithms, and accurately enough too for this Purpoſe. For the Log. of 2 (which is

0.3010300) multiplied by 64, is 19.2659200; the absolute Number agreeing to this, will be greater than 18446.00000.00000.00000 and less than 18447.00000.00000.00000."

Then by allowing 7680 Wheat Corns to a Statute Pint, and Wheat at 4 *s.* per Bushel, the Reader may find, at his Leisure, to what an immense Sum the Value of the above-mentioned Grains will arise.

CHAP. XXI.

The Valuation of Annuities upon Lives.

455. AS this is a very necessary Subject, we ought not to pass over it in this Place; and tho' it has been handled by learned Pens, yet, as some have given Rules and Solutions to Questions and Cases of this Sort, without any Theory or Demonstration at all; and others have given Theorems too tedious and perplexed; we shall here endeavour to steer a Course between both; and propose some of the most useful Cases, with as plain a Rationale thereof as we can.

456. And it is first to be observed, that before the Value of any Annuity on a Life of any proposed Age can be at all ascertained, it will be previously necessary to consider, how we are to estimate the Probability of the Continuance of a Life for any given Time; and it is evident this can be done no other Way but from Observations made from the *Bills of Mortality*, for a Series of Years together, and such as are proper to the Country where those Computations are to be made. For the *Bills of Mortality* at *Breslaw* will by no Means suit the Meridian of *London*; and therefore we shall first exhibit a Table of the Number of Persons who have died in every Year from 1728 to 1747 inclusive, or 20 Years, as taken from the yearly Bills of Mortality of this Metropolis, for the Ages mentioned at the Top of the respective Colums.

457. The

457. The Bills of Mortality alſo mention the Number of Perſons born in each Year; and from a Mean of the Whole, for 1000 Perſons born, there die the Numbers mentioned in the firſt Table at the Bottom, for the ſeveral Intervals of Age, as there ſpecified; where 'tis obſervable, that the leaſt Number die (*viz* 31 in 1000) between 10 and 20; and the greateſt Number (*viz.* 96 in 1000) between 40 and 50 Years of Age.

458. From the firſt Table the Second is conſtructed, where the Numbers of the ſeveral Intervals of Age are diſtributed into Numbers proper to the Years of thoſe Intervals; for Inſtance, between 10 and 20 Years there die 31 Perſons; but theſe in the following Table are divided and allotted to each ſingle Year as follows; 4, 4, 3, 2, 2, 2, 2, 3, 4, 5, the Sum of which makes 31. From hence the Probability of Living any propoſed Number of Years, at any Stage of Life, may be very eaſily deduced, as we ſhall ſhew.

459. TABLE

TABLE I.

459. *An Account of the Number of Persons that*
Twenty Years, taken from the Yearly Bills of

Years.	Under 2 Years of Age.	between 2 & 5	5 & 10	10 & 20	20 & 30	30 & 40
1728	9851	2407	1038	950	2254	2490
1729	10735	2516	1056	999	2371	2784
1730	10368	2448	1092	901	2048	2471
1731	9907	2096	932	806	1916	2351
1732	9502	1517	716	611	1627	2175
1733	11738	2409	957	754	1857	2564
1734	10752	2830	1228	829	1718	2212
1735	9672	1963	755	691	1605	2158
1736	10580	2706	993	816	2139	2445
1737	10054	2613	1008	885	2241	2652
1738	9600	2366	784	783	2072	2439
1739	9687	2302	844	875	1866	2218
1740	10765	2862	1235	947	2205	2783
1741	10456	2368	1072	1048	2816	3476
1742	9030	2642	1035	893	2203	2813
1743	8621	1955	947	813	1935	2342
1744	7394	1657	679	663	1744	2019
1745	7689	1631	672	626	1695	1940
1746	9503	2611	1089	895	2356	2728
1747	8741	2085	905	790	2190	2649
Totals of each Age.	194645	45984	19037	16575	40858	49709
$\frac{1}{20}$ Part for the mean Year.	9732	2299	952	829	2043	2485
1000 in the same Proportion.	368	87	36	31	77	94

TABLE I.

have died at the ſeveral Ages undermentioned, for Mortality of the City of London.

40 & 50	50 & 60	60 & 70	70 & 80	80 & 90	90 & 100	100 & upwards.	Totals of each Year.
2624	2123	1863	1290	785	125	10	27810
2698	2338	1938	1375	769	137	6	29722
2373	1713	1577	1001	622	138	9	26761
2261	1839	1500	913	628	108	5	25262
2121	1741	1581	974	660	121	12	23358
2685	2196	1871	1188	804	198	12	29233
2154	1668	1324	793	484	66	4	26062
2138	1684	1339	872	565	84	12	23538
2357	2121	1666	1114	557	83	4	27581
2578	2270	1650	1164	576	127	5	27823
2363	2106	1551	1121	529	101	10	25825
2378	2039	1421	1166	547	76	13	25432
2866	2585	1977	1716	758	100	12	30811
3731	2851	1933	1540	734	130	14	32169
2959	2333	1634	1250	582	100	9	27483
2611	2004	1729	1507	629	93	14	25200
2123	1637	1307	920	387	73	3	20606
2332	1741	1382	1064	437	77	10	21296
2876	2243	1699	1444	625	78	10	28157
2717	2079	1544	1199	520	68	7	25494
50945	41311	32486	23611	12198	2083	181	529623
2547	2066	1625	1181	610	104	9	26481
96	78	61	44	23	4		

460. TABLE

TABLE II.

460. *Exhibiting the Probabilities of Life, from the Bills of Mortality of the City of* London.

Years.	Number of Perfons dying off.	Number of Perfons living.	Years.	Number of Perfons dying off.	Number of Perfons living.	Years.	Number of Perfons dying off.	Number of Perfons living.	Years.	Number of Perfons dying off.	Number of Perfons living.
0		1000	24	7	452	48	9	229	72	5	61
1	276	724	25	8	444	49	9	220	73	5	56
2	92	632	26	8	436	50	9	211	74	5	51
3	49	583	27	8	428	51	9	202	75	5	46
4	25	558	28	9	419	52	9	193	76	4	42
5	13	545	29	9	410	53	8	185	77	4	38
6	10	535	30	9	401	54	8	177	78	4	34
7	8	527	31	9	392	55	8	169	79	4	30
8	7	520	32	9	383	56	8	161	80	3	27
9	6	514	33	9	374	57	7	154	81	3	24
10	5	509	34	9	365	58	7	147	82	3	21
11	4	505	35	9	356	59	7	140	83	3	18
12	4	501	36	9	347	60	7	133	84	3	15
13	3	498	37	10	337	61	7	126	85	2	13
14	2	496	38	10	327	62	7	119	86	2	11
15	2	494	39	10	317	63	6	113	87	2	9
16	2	492	40	10	307	64	6	107	88	2	7
17	2	490	41	10	297	65	6	101	89	2	5
18	3	487	42	10	287	66	6	95	90	1	4
19	4	483	43	10	277	67	6	89	91	1	3
20	5	478	44	10	267	68	6	83	92	1	2
21	6	472	45	10	257	69	6	77	93	1	1
22	6	466	46	10	247	70	5	71	94	1	0
23	7	459	47	9	238	71	5	66	95	0	

TABLE

TABLE III.

461. *Shewing the Probabilities of Life, from the Bills of Mortality, according to Mr.* SIMPSON.

Age.	N°. of Persons.	Age.	N°. of Persons.	Age.	N°. of Persons.	Age.	N°. of Persons.
born	1280	20	462	40	294	60	130
	—410		—7		—10		—7
1	870	21	455	41	284	61	123
	—170		—7		—10		—6
2	700	22	448	42	274	62	117
	—65		—7		—10		—6
3	635	23	441	43	264	63	111
	—35		—7		—9		—6
4	600	24	434	44	255	64	105
	—20		—8		—9		—6
5	580	25	426	45	246	65	99
	—16		—8		—9		—6
6	564	26	418	46	237	66	93
	—13		—8		—9		—6
7	551	27	410	47	228	67	87
	—10		—8		—8		—6
8	541	28	402	48	220	68	81
	—9		—8		—8		—6
9	532	29	394	49	212	69	75
	—8		—9		—8		—6
10	524	30	385	50	204	70	69
	—7		—9		—8		—5
11	517	31	376	51	196	71	64
	—7		—9		—8		—5
12	510	32	367	52	188	72	59
	—6		—9		—8		—5
13	504	33	358	53	180	73	54
	—6		—9		—8		—5
14	498	34	349	54	172	74	49
	—6		—9		—7		—4
15	492	35	340	55	165	75	45
	—6		—9		—7		—4
16	486	36	331	56	158	76	41
	—6		—9		—7		—3
17	480	37	322	57	151	77	38
	—6		—9		—7		—3
18	474	38	313	58	144	78	35
	—6		—9		—7		—3
19	468	39	304	59	137	79	32
	—6		—10		—7		—3
20	462	40	294	60	130	80	29

462. I have thought it proper to give both these Tables of the Probabilities of Life, but in the future Computations shall confine myself to that of Mr. *Simpson*, because his Tables of *Annuities on Lives*, will best suit my Purpose; but before any Thing on this Subject can be well understood, we must premise the following lemmatical Propositions.

463. *The Probability of the happening of any Event is in Proportion to the Chances which that Event has to happen to the Number of all the Chances which it has both to happen and to fail.* Thus suppose *one Die* was to be cast, there is but *one Chance* that any one Number of Spots on it shall come up; and it is evident there are five Chances for it to fail, therefore the Probability that any particular Number shall come up, will be as 1 to 1 + 5, that is, as 1 to 6; and may be thus expressed, $\frac{1}{6}$. Thus if the *Octahedron*, or Body of eight Sides, had the eight Digits on them respectively, then the Probability of any one coming up on a Cast, would be as $\frac{1}{8}$.

464. Therefore, if the *Die* and *Octahedron* were both cast up together, the Probability that any two Numbers you shall name, come up together, will be as $\frac{1}{6} \times \frac{1}{8}$, or $\frac{1}{48}$. Again, if a *Dodecahedron*, or Body of 12 Sides, were added to the two former, with the proper Numbers on it; and all three were thrown together, the Probability that any three Numbers specified, come up together, will be, as $\frac{1}{6} \times \frac{1}{8} \times \frac{1}{12}$, or $\frac{1}{576}$; and so of any other Number.

465. Hence, if a Sum of Money was to be expected on the coming up of any Number on the Die, 'tis plain the Value of such an Expectation would be but a $\frac{1}{6}$ Part of that Sum; or if the receiving a Sum depends on the happening of any two Numbers on the *Die* and *Octahedron* together, the Value of the Expectation would be but $\frac{1}{48}$ Part of that Sum.

466. If, in the second of the foregoing Tables, we find at the Age of 25 Years there are 444 People living out of 1000, and against the Age of 60, there are but 133, it appears that in that Interval there have died 311: So that the Number of Chances which a Person of 25 Years of Age has to live to 60, will be as 133, and the Chances he has to fail will be as 311; therefore his Probability of living to the Age of 60 Years will be as 133 to 133 + 321, or as $\frac{133}{444}$. (by 463.)

467. Let

467. Let $a =$ Number of Chances any Event has to happen, and $b =$ the Number of all the Chances there are for it to happen and fail; and put $p =$ to the Probability of it happening; then by what we have premised, we have this Theorem, $p = \frac{a}{b}$, therefore $pb = a = a \times 1$; whence we have $1 : p :: b : a$; and then $1 - p : p :: b - a : a$ (as will appear by multiplying Extremes and Means.) In the above Example, $a = 133$, $b = 444$, and $p = \frac{a}{b} = 0{,}3$. The Chance therefore a Person of 25 has to live to 60 is as 0,3 to 1, and not as 0,4276 to 1, as some have asserted.

468. If any Sum S be expected on the happening of any Event, the Value V of that Expectation will be in Proportion to the Sum depending, and the Probability of the Event's happening, that is, V will be as p S; therefore $V : pS : \frac{aS}{b}$; and so we have $bV = aS$; and $V : S :: a : b$.

469. Suppose a Person A was to receive 100 *l.* upon this Condition, that another, B, of 20 Years Age, should live one Year; Query the Value of A's Expectation? In Table III. (for we shall use Mr. *Simpson*'s Numbers for the future) against the Age of 20 and 21, the Numbers are 462, and 455; whence $a : b :: 455 : 462$. Now the present Value S of 100 *l.* due at the End of one Year, allowing (suppose) 4 *l. per Cent.* is 96,15 *l.* (See Table III. p. 179.) Wherefore $V = \frac{aS}{b} = \frac{455 \times 96.15}{462}$ $= 94.7$ *l.* the true Value required.

470. In like Manner the Probability a Person of 20 Years of Age has of living 2 Years is $\frac{448}{462}$; and the present Worth of 100 *l.* due at the End of 2 Years (at 4 *per Cent.*) is 92,45; therefore $\frac{448}{462} \times 92{,}45 = 89{,}65$ *l.* the Value of A's Expectation to receive 100 *l.* at the End of the 22d Year of B's Life. And thus you proceed for all the other Years of his Life to the Extremity of Age; and the Sum of all these being found, and added together, will amount to 1480 *l.* very nearly. But if Table II. of Mr. *Stonehouse* be used, then the Value of an Annuity for such a Life will amount to 1485 *l.*

 471. Since

471. Since $\frac{A}{R^t} = P$ (402.) and putting $t = 1, 2, 3, 4, 5,$ &c. and $A = 1\,l.$ 'tis plain the Sum of the Values of $1\,l.$ for the successive Years of a Person's Life will be expressed by this Series $\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \frac{1}{R^4}$, &c. were there no Contingency in the Case. But as we must allow for that; let Q be the Number in the Table, corresponding to the given Age of B, and $\dot{Q}.\ \ddot{Q}.\ \dddot{Q}.$ &c. be those answering to the next succeeding Ages respectively; then we shall have $\frac{\dot{Q}}{QR} + \frac{\ddot{Q}}{QR^2} + \frac{\dddot{Q}}{QR^3}$ &c. or $\frac{1}{Q} \times \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2} + \frac{\dddot{Q}}{R^3}$, &c. equal to the Value of an Annuity of $1\,l.$ on the Life of B.

472. When the Value (V) of any one Life is computed or given, the Value (v) of the next younger Life will be easily deduced from thence; for let $\dot{q}$ be the Number in the Table found against the next younger Age; then (by 471.) for the same Reason that $V = \frac{1}{Q} \times \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2} + \frac{\dddot{Q}}{R^3}$, &c. will $v = \frac{Q}{R\dot{q}} + \frac{\dot{Q}}{R^2\dot{q}} + \frac{\ddot{Q}}{R^3\dot{q}}$, &c. $= \frac{1}{R\dot{q}} \times Q + \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2}$, &c. Whence 'tis plain, by multiplying the former Series by Q, and the latter by $R\dot{q}$, we get $QV = \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2} + \frac{\dddot{Q}}{R^3}$, &c. and $vR\dot{q} = Q + \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2}$, &c. Therefore $vR\dot{q} - Q = QV$; and so $QV + Q = vR\dot{q}$; consequently $v = \frac{QV + Q}{R\dot{q}} = \overline{V + 1} \times \frac{Q}{R\dot{q}} = \overline{V + 1} \times \frac{1}{R} \times \frac{Q}{\dot{q}}$.

473. This last Theorem is thus expressed in Words; To the Value of the given Life (V) add one Year's Purchase, and multiply

tiply that Sum $(V + 1)$ by $\frac{1}{R}$ (which is the ſame Thing as to diſcount it for one Year,) and that Product again multiply by the Probability of a Life of *Nineteen* continuing one Year (*viz.* $\frac{Q}{q}$) ; this laſt Product will be the Value (V) of an Annuity upon this Liſe. For Example, The Life of 20 being 1480 *l.* this encreaſed by one Year's Purchaſe is 1580*l.* which diſcounted at 4 *per Cent.* is 1519.2; this multiplied by the Probability of a Life of Nineteen, *viz.* $\frac{462}{468}$ gives 1499,8, or 1500*l.* for the required Value of ſuch a Life.

474. If the Annuity was but 1 *l.* inſtead of 100*l.* the Value of ſuch an Annuity on a Life of 20, would have been 14,8 *l.* and for a Life of 19, it would be 15 *l.* and in ſuch a Manner are the Numbers in the following Table IV. computed, which ſhews the Value of an Annuity on any ſingle Life from ſix Years of Age to 75 (according to Mr. *Simpſon*,) at the Rate of 3, 4, 5, *per Cent.*

N. B. Compound Intereſt is always allowed in theſe Caſes. And the Value of an Annuity on any Life, is called ſuch a Number of *Years Purchaſe*, as it is the preſent Value of the Annuity to continue for ſo many Years certain. Thus the preſent Value of an Annuity of 100 *l.* *per Annum* on a Life of 20 Years of Age, diſcounting 4 *per Cent.* is equal to $14{,}8 \times 100 =$ 1480 *l.* as per Calculation (470.)

TABLE

TABLE II.

For the Valuation of Annuities upon one LIFE.

Age.	Year's Purch. at 5 *per Cent.*	Year's Purch. at 4 *per Cent.*	Year's Purch. at 3 *per Cent*	Age.	Year's Purch. at 5 *per Cent*	Year's Purch at 4 *per Cent*	Year's Purch at 3 *per Cent.*	Age.	Year's Purch. at 5 *per Cent.*	Year's Purch. at 4 *per Cent.*	Year's Purch at 3 *per Cent*
6	14.1	16.2	18.8	31	11.4	12.9	14.8	56	8.4	9.1	10.1
7	14.2	16.3	18.9	32	11.3	12.7	14.6	57	8.2	8.9	9.9
8	14.3	16.4	19.0	33	11.2	12.6	14.4	58	8.1	8.7	9.6
9	14.3	16.4	19.0	34	11.0	12.4	14.2	59	8.0	8.6	9.4
10	14.3	16.4	19.0	35	10.9	12.3	14.1	60	7.9	8.4	9.2
11	14.3	16.4	19.0	36	10.8	12.1	13.9	61	7.7	8.2	8.9
12	14.2	16.3	18.9	37	10.6	11.9	13.7	62	7.6	8.1	8.7
13	14.1	16.2	18.7	38	10.5	11.8	13.5	63	7.4	7.9	8.5
14	14.0	16.0	18.5	39	10.4	11.6	13.3	64	7.3	7.7	8.3
15	13.9	15.8	18.3	40	10.3	11.5	13.2	65	7.1	7.5	8.0
16	13.7	15.6	18.1	41	10.2	11.4	13.0	66	6.9	7.3	7.8
17	13.5	15.4	17.9	42	10.1	11.2	12.8	67	6.7	7.1	7.6
18	13.4	15.2	17.6	43	10.0	11.1	12.6	68	6.6	6.9	7.4
19	13.2	15.0	17.4	44	9.9	11.0	12.5	69	6.4	6.7	7.1
20	13.0	14.8	17.2	45	9.8	10.8	12.3	70	6.2	6.5	6.9
21	12.9	14.7	17.0	46	9.7	10.7	12.1	71	6.0	6.3	6.7
22	12.7	14.5	16.8	47	9.5	10.5	11.9	72	5.8	6.1	6.5
23	12.6	14.3	16.5	48	9.4	10.4	11.8	73	5.6	5.9	6.2
24	12.4	14.1	16.3	49	9.3	10.2	11.6	74	5.4	5.6	5.9
25	12.3	14.0	16.1	50	9.2	10.1	11.4	75	5.2	5.4	5.6
26	12.1	13.8	15.9	51	9.0	9.9	11.2				
27	12.0	13.6	15.6	52	8.9	9.8	11.0				
28	11.8	13.4	15.4	53	8.8	9.6	10.7				
29	11.7	13.2	15.2	54	8.6	9.4	10.5				
30	11.6	13.1	15.0	55	8.5	9.3	10.3				

476. We

476. We might now proceed to compute the Value of Annuities on two, three, or more *joint* Lives, or on the longest Liver of them, on the same Principles as before; but we shall desist for the present, from any further Investigations of this Kind, for two Reasons; first, because of the great Difficulty and Intricacy of them; and, second, because they are not so necessary, or so often in Use as Annuities on a single Life. And to say, what I think is Truth, the whole Affair of *buying and selling Annuities* on several Lives, seems to be a Sort of *gaming*, at least a Matter of great Uncertainty after all; for general Rules deduced from the best Bills of Mortality that could be had, must be very fallacious, and short of ascertaining the present real Value of an Annuity on any particular Life or Lives, and can be looked upon only as affording *Mediums* and *Approximations* thereto. As we have shewn the general Nature of this *Calculus*, and given the most useful Part, we shall next proceed to what remains of the *Algebraic Institutions*.

CHAP. XXII.

Of the Nature, Genesis, and Roots of Cubic Equations, and those of higher Dimensions.

477. AS we have been sufficiently prolix on the Nature and Genesis of *Quadratic Equations* (in Chap. XIV.) the less will be necessary to be said here in regard to *Cubic Ones*, and those of higher Powers. For the Nature and Rationale of all depend on that of the *component, or generating Roots*. Thus, suppose the Value of the unknown Quantity x in any Equation were to be expressed by a, b, c, d, &c. that is, let $x = a$, $x = b$, $x = c$, $x = d$, &c. then will $x - a = 0$, $x - b = 0$, $x - c = 0$, $x - d = 0$, &c. be the simple *radical Equations*, of which those of higher Orders are composed; and as the Product of any two of these gives a *Quadratic Equation*, or one of two Dimensions; so the Product of any three of them as $\overline{x - a} \times \overline{x - b} \times \overline{x - c} = 0$, will give a *Cubic Equation* or one of three

three Dimensions. And the Product of four of them will constitute a *Biquadratic Equation* or one of four Dimensions; and so on. Therefore, in general, *the highest Dimension of the unknown Quantity x is equal to the Number of simple Equations that are multiplied together to produce it.*

478. When any Equation equivalent to this Biquadratic $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$ is proposed to be resolved, the whole Difficulty consists in finding the simple Equations $\overline{x-a}=0$, $\overline{x-b}=0$, $\overline{x-c}=0$, $\overline{x-d}=0$, by whose Multiplication it is produced; for each of these simple Equations gives one of the Values of x, and one Solution of the proposed Equation. For, if any of the Values of x deduced from those simple Equations be substituted in the proposed Equation, in place of x, then all the Terms of that Equation will vanish, and the whole be found equal to nothing. Because when it is supposed that $x=a$, or $x=b$, or $x=c$, or $x=d$, then the Product $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}$ does vanish, because one of the Factors is equal to nothing. There are therefore four Suppositions that give $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$ according to the proposed Equation; that is, there are four Roots of the proposed Equation. And after the same Manner *any other* Equation admits of as many Solutions as there are simple Equations multiplied by one another that produce it, or as many as there are Units in the highest Dimension of the unknown Quantity in the proposed Equation.

479. But as there are no other Quantities whatsoever besides these four (a, b, c, d,) that substituted in the Product $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}$, in the Place of x, will make the Product vanish; therefore, the Equation $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$, cannot possibly have more than these four Roots, and cannot admit of more Solutions than four. If you substitute in that Product a Quantity neither equal to a, nor b, nor c, nor d, which suppose e, then since neither, $e-a$, $e-b$, $e-c$, nor $e-d$ is equal to nothing; their Product $\overline{e-a}\times\overline{e-b}\times\overline{e-c}\times\overline{e-d}$ cannot be equal to nothing, but must be some real Product: And therefore, there is no Supposition

position beside one of the foresaid Four, that gives a just Value of x according to the proposed Equation. So that it can have no more than these four Roots. And after the same Manner it appears, that *no Equation can have more Roots than it contains Dimensions of the unknown Quantity.*

480. To make all this still plainer by an Example, in Numbers; suppose the Equation to be resolved to be $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, and that you discover that this Equation is the same with the Product of $\overline{x-1} \times \overline{x-2} \times \overline{x-3} \times \overline{x-4}$, then you certainly infer that the four Values of x are 1, 2, 3, 4; seeing any of these Numbers, placed for x, makes that Product, and consequently $x^4 - 10x^3 + 35x^2 - 50x + 24$, equal to nothing, according to the proposed Equation. And it is certain that there can be no other Values of x besides these four: Since, when you substitute any other Number for x in those Factors $x-1$, $x-2$, $x-3$, $x-4$, none of the Factors vanish, and therefore their Product cannot be equal to nothing, according to the Equation.

481. *The Number of Terms is always greater than the highest Dimension of the unknown Quantity by Unit.* And when any Term is wanting, an *Asterisk* is marked in its Place. The *Signs* and *Coefficients* of Equations will be understood by considering the following Table, where the simple Equations $x-a$, $x-b$, &c. are multiplied by one another, and produce, successively, the higher Equations.

$$x - a = 0$$
$$\times\ \overline{x - b} = 0$$
$$= \left.\begin{array}{l} x^2 - ax \\ \quad - bx + ab \end{array}\right\} = 0,\ \text{a } \textit{Quadratic.}$$
$$\times\ \overline{x - c} = 0$$
$$= x^3 \left.\begin{array}{l} -a \\ -b \\ -c \end{array}\right\} \times x^2 \left.\begin{array}{l} +ab \\ +ac \\ +bc \end{array}\right\} \times x - abc = 0,\ \text{a } \textit{Cubic.}$$
$$\times\ \overline{x - d} = 0$$
$$= x^4 \left.\begin{array}{l} -a \\ -b \\ -c \\ -d \end{array}\right\} \times x^3 \left.\begin{array}{l} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} \times x^2 \left.\begin{array}{l} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} \times x + abcd = 0,\ \text{a } \textit{Biquadratic.}$$
$$\times\ \overline{x - e} = 0$$
$$= x^5 \left.\begin{array}{l} -a \\ -b \\ -c \\ -d \\ -e \end{array}\right\} \times x^4 \left.\begin{array}{l} +ab \\ +ac \\ +ad \\ +ae \\ +bc \\ +bd \\ +be \\ +cd \\ +ce \\ +de \end{array}\right\} \times x^3 \left.\begin{array}{l} -abc \\ -abd \\ -abe \\ -acd \\ -ade \\ -ace \\ -bcd \\ -bce \\ -bde \\ -cde \end{array}\right\} \times x^2 \left.\begin{array}{l} +abcd \\ +abce \\ +abde \\ +acde \\ +bcde \end{array}\right\} + x - abcde = 0.$$

(a *Surfolid.*)

&c.

482. From the Inſpection of theſe Equations it is plain, that the Coefficients of the firſt Term is *Unit.*

The Coefficient of the ſecond Term is *the Sum of all the Roots* (a, b, c, d, e,) *having their Signs changed.*

The Coefficient of the third Term is *the Sum of all the Product, that can be made by multiplying any two of the Roots* (a, b, c, d, e,) *by one another.*

The Coefficient of the fourth Term is *the Sum of all the Products that can be made by multiplying into one another any three of the Roots, with their Signs changed.* And after the ſame Manner all the other Coefficients are formed.

The laſt Term is always *the Product of all the Roots having their Signs changed,* multiplied by one another.

483. Altho'

483. Altho' in the Table such simple Equations only are multiplied by one another as have positive Roots, it is easy to see, "that the Coefficients will be formed according to the same Rule when any of the simple Equations have negative Roots." And, *in general*, if $x^3 - px^2 + qx - r = 0$ represent any Cubic Equation, then shall p be the Sum of the Roots; q the Sum of the Products made by multiplying any two of them; r the Product of all the three: And, if $-p$, $+q$, $-r$, $+s$, $-t$, $+u$, &c. be the Coefficients of the 2d, 3d, 4th, 5th, 6th, 7th, *&c.* Terms of any Equation, then shall p be the Sum of all the Roots, q the Sum of the Products of any two, r the Sum of the Products of any Three, s the Sum of the Products of any Four, t the Sum of the Products of any Five, u the Sum of the Products of any Six, *&c.*

484. When therefore any Equation is proposed to be resolved, it is easy to find the Sum of the Roots, (for it is equal to the Coefficient of the second Term having its Sign changed:) or, to find the Sum of the Products that can be made by multiplying any determinate Number of them.

485. But it is also easy "to find the Sum of the Squares, or of any Powers, of the Roots."

The Sum of the Squares is always $p^2 - 2q$. For calling the Sum of the Squares B, since the Sum of the Roots is p; and "the Square of the Sum of any Quantities is always equal to the Sum of their Squares added to double the Products that can be made by multiplying any two of them," therefore, $p^2 = B + 2q$, and consequently, $B = p^2 - 2q$. For Example, $\overline{a+b+c}|^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. That is $p^2 = B + 2q$. And $\overline{a+b+c+d}|^2 = a^2 + b^2 + c^2 + d^2 + 2 \times \overline{ab + ac + ad + bc + bd + cd}$, that is again $p^2 = B + 2q$, or $B = p^2 - 2q$. And so for any other Number of Quantities. In general therefore, "B the Sum of the Squares of the Roots may always be found by subtracting $2q$ from p^2;" the Quantities p and q being always known, since they are the Coefficients in the proposed Equation.

486. "The Sum of the Cubes of the Roots of any Equation is equal to $p^3 - 3pq + 3r$, or to $Bp - pq + 3r$." For $\overline{B - q} \times p$ gives always the Excess of the Sum of the Cubes of any Quantities above the triple Sum of the Products

 that

that can be made by multiplying any three of them. Thus $\overline{a^2 + b^2 + c^2 - ab - ac - bc} \times \overline{a + b + c}$ $(= \overline{B - q} \times p) = a^3 + b^3 + c^3 - 3abc$. Therefore, if the Sum of the Cubes is called C, then ſhall $\overline{B - q} \times p = C - 3r$, and $C = Bp - qp + 3r =$ (becauſe $B = p^2 - 2q$) $= p^3 - 3pq + 3r$.

After the ſame Manner, if D be the Sum of the 4th Powers of the Roots, you will find that $D = pC - qB + pr - 4s$, and if E be the Sum of the 5th Powers, then ſhall $E = pD - qC + rB - ps + 5t$. And after the ſame Manner the Sum of any Powers of the Roots may be found; the Progreſſion of theſe Expreſſions of the Sum of the Powers being obvious.

487. As for the Signs of the Terms of the Equation produced, it appears from Inſpection that the Signs of all the Terms in any Equation in the Table are alternately + and —: Theſe Equations are generated by multiplying continually $x - a$, $x - b$, $x - c$, $x - d$, &c. by one another. The firſt Term is always ſome Pure Power of x, and is poſitive; the ſecond is a Power of x multiplied by the Quantities $-a$, $-b$, $-c$, &c. And ſince theſe are all negative, that Term muſt therefore be negative. The third Term has the Products of any two of theſe Quantities ($-a$, $-b$, $-c$, &c.) for its Coefficient; which Products are all poſitive, becauſe $- \times -$ gives +. For the like Reaſon, the next Coefficient, conſiſting of all the Products made by multiplying any three of theſe Quantities, muſt be negative: And the next poſitive. So that the Coefficients in this Caſe, will be poſitive and negative by Turns. But "in this Caſe, the Roots are all poſitive;" ſince $x = a$, $x = b$, $x = c$, $x = d$, $x = e$, &c. are the aſſumed ſimple Equations. It is plain then, that, "*when all the Roots are poſitive, the Signs are alternately + and —.*"

488. But if the Roots are all negative, then $\overline{x + a} \times \overline{x + b} \times \overline{x + c} \times \overline{x + d}$, &c. $= 0$ will expreſs the Equation to be produced; all whoſe Terms will plainly be poſitive; ſo that "*when all the Roots of an Equation are negative, it is plain there will be no changes in the Signs of the Terms of that Equation.*"

489. In general, "there are as many poſitive Roots in any Equation as there are Changes in the Signs of the Terms from

+ to

$+$ to $-$, or from $-$ to $+$; and the remaining Roots are negative." The Rule is general, if the impossible Roots be allowed to be either positive or negative.

490. In Quadratic Equations, the two Roots are either both positive, as in this

$$(\overline{x-a} \times \overline{x-b} =)\ x^2 \begin{matrix} -ax \\ -bx \end{matrix} + ab = 0,$$

where there are two Changes of the Signs: Or, they are both negative as in this

$$(\overline{x+a} \times \overline{x+b} =)\ x^2 \left.\begin{matrix} +a \\ +b \end{matrix}\right\} x + ab = 0,$$

where there is not any Change of the Signs. Or there is one positive and one negative, as in

$$(\overline{x-a} \times \overline{x+b} =)\ x^2 \left.\begin{matrix} -a \\ +b \end{matrix}\right\} x - ab = 0,$$

where there is necessarily one Change of the Signs; because the first Term is positive, and the last negative, and there can be but one Change whether the 2d Term be $+$ or $-$.

Therefore the Rule given (Inst. 489.) extends to all Quadratic Equations.

491. In Cubic Equations, the Roots may be,

1°. All positive as in this, $\overline{x-a} \times \overline{x-b} \times \overline{x-c} = 0$, in which the Signs are alternately $+$ and $-$, as appears from the Table; and there are three Changes of the Signs.

2°. The Roots may be all negative as in the Equation $\overline{x+a} \times \overline{x+b} \times \overline{x+c} = 0$, where there can be no Change of the Signs. Or,

3°. There may be two positive Roots and one negative, as in the Equation $\overline{x-a} \times \overline{x-b} \times \overline{x+c} = 0$; which gives

$$x^3 \left.\begin{matrix} -a \\ -b \\ +c \end{matrix}\right\} x^2 \left.\begin{matrix} +ab \\ -ac \\ -bc \end{matrix}\right\} x + abc = 0.$$

Here

Here there muſt be two Changes of the Signs, becauſe if $a + b$ is greater than c, the ſecond Term muſt be negative, its Coefficient being $- a - b + c$.

And if $a + b$ is leſs than c, then the third Term muſt be negative, its Coefficient $+ ab - ac - bc$ $(ab - c \times \overline{a + b})$* being in that Caſe negative. And there cannot poſibly be three Changes of the Signs, the firſt and laſt Terms having the ſame Sign.

492. 4°. There may be one poſitive Root and two negative, as in the Equation $\overline{x + a} \times \overline{x + b} \times \overline{x - c} = 0$, which gives

$$x^3 \left.\begin{matrix} + a \\ + b \\ - c \end{matrix}\right\} x^2 \left.\begin{matrix} + ab \\ - ac \\ - bc \end{matrix}\right\} x - abc = 0.$$

Where there muſt be always one Change of the Signs, ſince the firſt Term is poſitive and the laſt negative. And, there can be but one Change of the Signs, ſince if the ſecond Term is negative, or $a + b$ leſs than c, the third muſt be negative alſo, ſo that there will be but one Change of the Signs. Or, if the ſecond Term is affirmative, whatever the third Term is, there will be but one Change of the Signs. It appears therefore, in general, that in Cubic Equations, there are as many affirmative Roots as there are Changes of the Signs of the Terms of the Equation.

The ſame Way of Reaſoning may be extended to Equations of higher Dimenſions, and the Rule delivered in Inſt. 489. extended to all Kinds of Equations.

493. There are ſeveral Conſectaries of what has been already demonſtrated, that are of Uſe in diſcovering the Roots of Equations. But before we proceed to that, it will be convenient to explain ſome Transformations of Equations, by which they may often be rendered more ſimple, and the Inveſtigation of their Roots more eaſy.

CHAP.

* Becauſe the Rectangle $a \times b$ is leſs than the Square $\overline{a + b} \times \overline{a + b}$, and therefore much leſs than $\overline{a + b} \times c$.

CHAP. XXIII.

Of the Transformation of Equations; and exterminating their intermediate Terms.

494. WE now proceed to explain the Transformations of Equations that are most useful: And first, *The Affirmative Roots of an Equation are changed into negative Roots of the same Value, and the negative Roots into affirmative, by only changing the Signs of the Terms alternately, beginning with the Second.* Thus, the Roots of the Equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ are $+1$, $+2$, $+3$, -5; whereas the Roots of the same Equation having only the Signs of the second and fourth Terms changed, *viz.* $x^4 + x^3 - 19x^2 - 49x^2 - 30 = 0$ are -1, -2, -3, $+5$.

To understand the Reason of this Rule, let us assume an Equation, as $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e}$, &c. $= 0$, whose Roots are $+a$, $+b$, $+c$, $+d$, $+e$, &c. and another having its Roots of the same Value, but affected with contrary Signs, as $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e}$, &c. $= 0$. It is plain, that the Terms taken alternately, beginning from the first,* are the same in both Equations, and have the same Sign, "being Products of an even Number of the Roots;" the Product of any two Roots having the same Sign as their Product when both their Signs are changed; as $+a \times -b = -a \times +b$.

But the second Terms, and all taken alternately from them, because their Coefficients involve always the Products of an odd Number of the Roots, will have contrary Signs in the two Equations. For Example, the Product of four *viz.* $abcd$, having the same Sign in both, and one Equation in the fifth Term having $abcd \times +e$, and the other $abcd \times -e$, it follows, that their Product $abcde$ must have contrary Signs in the two Equations: These two Equations therefore that have the same Roots, but with contrary Signs, have nothing different but the Signs of the alternate Terms, beginning with the second. From which it follows, "that if any Equation is given and

* See the Table in Page 238.

and you change the Signs of the alternate Terms, beginning with the ſecond, the new Equation will have Roots of the ſame Value, but with contrary Signs."

495. It is often very uſeful "*to transform an Equation into another that ſhall have its Roots greater or leſs than the Roots of the propoſed Equation by ſome given Difference.*"

Let the Equation propoſed be the Cubic $x^3 - px^2 + qx - r = 0$. And let it be required to transform it into another Equation whoſe Roots ſhall be leſs than the Roots of this Equation by ſome given Difference (e), that is, ſuppoſe $y = x - e$, and conſequently $x = y + e$; then inſtead of x and its Powers, ſubſtitute $y + e$ and its Powers, and there will ariſe this new Equation

$$(A)\quad \left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array}\right\} = 0$$

whoſe Roots are leſs than the Roots of the preceding Equation by the Difference (e).

If it had been required to find an Equation whoſe Roots ſhould be greater than thoſe of the propoſed Equation by the Quantity (e), then we muſt have ſuppoſed $y = x + e$, and conſequently $x = y - e$, and then the other Equation would have had this Form

$$(B)\quad \left.\begin{array}{r} y^3 - 3ey^2 + 3e^2y - e^3 \\ - py^2 + 2pey - pe^2 \\ + qy - qe \\ - r \end{array}\right\} = 0$$

If the propoſed Equation be in this Form $x^3 + px^2 + qx + r = 0$, then by ſuppoſing $x + e = y$ there will ariſe an Equation agreeing in all Reſpects with the Equation (A), but that the ſecond and fourth Terms will have contrary Signs.

And by ſuppoſing $x - e = y$, there will ariſe an Equation agreeing with (B) in all Reſpects, but that the ſecond and fourth Terms will have contrary Signs to what they have in (B).

The firſt of theſe Suppoſitions gives this Equation,

(C)

$$(C)\ \left.\begin{array}{r} y^3 - 3ey^2 + 3e^2y - e^3 \\ + py^2 - 2pey + pe^2 \\ + qy - qe \\ + r \end{array}\right\} = 0$$

The ſecond Suppoſition gives the Equation,

$$(D)\ \left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ + py^2 + 2epy + pe^2 \\ + qy + qe \\ + r \end{array}\right\} = 0$$

496 The firſt Uſe of this Transformation of Equations is to ſhew *how the ſecond (or other intermediate) Term may be taken away out of an Equation.*

It is plain that in the Equation *(A)* whoſe ſecond Term is $\overline{3e - p} \times y^2$, if you ſuppoſe $e = \frac{1}{3}p$, and conſequently $3e - p = 0$, then the ſecond Term will vaniſh.

In the Equation *(C)* whoſe ſecond Term is $-\overline{3e + p} \times y^2$ ſuppoſing $e = \frac{1}{3}p$, the ſecond Term alſo vaniſhes.

Now the Equation *(A)* was deduced from $x^3 - px^2 + qx - r = 0$, by ſuppoſing $y = x - e$: And the Equation *(C)* was deduced from $x^3 + px^2 + qx + r = 0$; by ſuppoſing $y = x + e$; From which this Rule may eaſily be deduced for exterminating the ſecond Term out of any Cubic Equation.

RULE.

497. *Add to the unknown Quantity of the given Equation the third Part of the Coefficient of the ſecond Term with its proper Sign,* viz $\mp \frac{1}{3}p$, *and ſuppoſe this Aggregate equal to a new unknown Quantity* (y). *From this Value of* y *find a Value of* x *by Tranſpoſition, and ſubſtitute this Value of* x *and its Powers in the given Equation, and there will ariſe a new Equation that ſhall want the ſecond Term.*

EXAMPLE.

Let it be required to exterminate the ſecond Term out of this Equation, $x^3 - 9x^2 + 26x - 34 = 0$, ſuppoſe $x - 3 = y$, or $y + 3 = x$; and ſubſtituting according to the Rule, you will find

$$\left.\begin{array}{r} y^3 + 9y^2 + 27y + 27 \\ - 9y^2 - 54y - 81 \\ + 26y + 78 \\ - 34 \end{array}\right\} = 0$$

$$y^3 \quad * \quad - y - 10 = 0.$$

In which there is no Term where y is of two Dimenſions, and an Aſteriſk is placed in the room of the ſecond Term, to ſhew it is wanting.

498. Let the Equation propoſed be of any Number of Dimenſions repreſented by (n); and let the Coefficient of the ſecond Term with its Sign prefixed be $-p$, then ſuppoſing $x - \frac{p}{n} = y$, and conſequently $x = y + \frac{p}{n}$, and ſubſtituting this Value for x in the given Equation, there will ariſe a new Equation, that ſhall want the ſecond Term.

It is plain from what was demonſtrated in *Chap.* 2. that the Sum of the Roots of the propoſed Equation is $+ p$, and ſince we ſuppoſe $y = x - \frac{p}{n}$, it follows, that, in the new Equation, each Value of y will be leſs than the reſpective Value of x by $\frac{p}{n}$; and, ſince the Number of the Roots is n, it follows that the Sum of the Values of y will be leſs than $+ p$, the Sum of the Values of x, by $n \times \frac{p}{n}$, that is, by $+ p$: Therefore the Sum of the Values of y will be $+ p - p = 0$.

But the Coefficient of the ſecond Term of the Equation of y is the Sum of the Values of y, *viz.* $+ p - p$, and therefore that Coefficient is equal to nothing; and conſequently, in the Equation of y, the ſecond Term vaniſhes. It follows then, that the ſecond Term may be exterminated out of any given Equation by the following

RULE.

Divide the Coefficient of the ſecond Term of the propoſed Equation by the Number of Dimenſions of the Equation; and aſſuming a new unknown Quantity y, *add to it the Quotient having its Sign changed. Then*

Then ſuppoſe this Aggregate equal to x *the unknown Quantity in the propoſed Equation; and for* x *and its Powers, ſubſtitute the Aggregate and its Powers, ſo that the new Equation that ariſes want its ſecond Term.*

499. If the propoſed Equation is a Quadratic, as $x^2 - px + q = 0$, then, according to the Rule, ſuppoſe $y + \frac{1}{2}p = x$, and ſubſtituting this Value for x, you will find,

$$\left.\begin{array}{r} y^2 + py + \frac{1}{2}p^2 \\ - py - \frac{1}{2}p^2 \\ + \quad q \end{array}\right\} = 0$$

$$\overline{y^2 \quad * \quad - \tfrac{1}{4}p^2 + q = 0.}$$

And from this Example the Uſe of exterminating the 2d Term appears: For commonly the Solution of the Equation that wants the 2d Term is more eaſy. And, if you can find the Value of y from this new Equation, it is eaſy to find the Value of x by Means of the Equation $y + \frac{1}{2}p = x$. For Example,

Since $y^2 + q - \frac{1}{4}p^2 = 0$, it follows that

$y^2 = \frac{1}{4}p^2 - q$, and $y = \pm\sqrt{\frac{1}{4}p^2 - q}$, ſo that $x = y + \frac{1}{2}p = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$;

which agrees with what we demonſtrated (Inſt. 339.)

If the propoſed Equation is a Biquadratic, as $x^4 - px^3 + qx^2 - rx + s = 0$, then by ſuppoſing $x - \frac{1}{4}p = y$ or $x = y + \frac{1}{4}p$, an Equation ſhall ariſe having no ſecond Term. And if the propoſed is of five Dimenſions, then you muſt ſuppoſe $x = y \pm \frac{1}{5}p$. And ſo on.

500. When the ſecond Term in any Equation is wanting, it follows, that "the Equation has both affirmative and negative Roots," and that the Sum of the affirmative Roots is equal to the Sum of the negative Roots." By which Means the Coefficient of the ſecond Term, which is the Sum of all the Roots of both Sorts, vaniſhes, and makes the ſecond Term vaniſh.

In general, "the Coefficient of the ſecond Term is the Difference between the Sum of the Affirmative Roots and the Sum of the negative Roots:" And the Operations we have given ſerve only to diminiſh all the Roots when the Sum of the Af-

firmative is greatest, or increase the Roots when the Sum of the Negative is greatest, so as to balance them, and reduce them to an Equality.

It is obvious, that in a Quadratic Equation that wants the second Term, there must be one Root affirmative and one negative; and these must be equal to one another.

In a Cubic Equation that wants the second Term, there must be either two affirmative Roots equal, taken together, to a third Root that must be negative; or two negative equal to a third that must be positive.

501. Let an Equation $x^3 - px^2 + qx - r = 0$ be proposed, and let it be now required to exterminate the third Term.

By supposing $y = x - e$, the Coefficient of the third Term in the Equation of y is found (see Equation A) to be $3e^2 - 2pe + q$. Suppose that Coefficient equal to nothing, and by resolving the Quadratic Equation $3e^2 - 2pe + q = 0$, you will find the Value of e, which substituted for it in the Equation $y = x - e$, will shew how to transform the proposed Equation into one that shall want the third Term.

The Quadratic $3e^2 - 2pe + q = 0$ gives $e = \frac{p \pm \sqrt{p^2 - 3q}}{3}$.
So that the proposed Cubic will be transformed into an Equation wanting the third Term by supposing $y = x - \frac{p - \sqrt{p^2 - 3q}}{3}$, or $y = x - \frac{p + \sqrt{p^2 - 3q}}{3}$.

502. If the proposed Equation is of n Dimensions, the Value of e by which the third Term may be taken away, is had by resolving the Quadratic Equation $e^2 + \frac{2p}{n} \times e + \frac{2q}{n \times \overline{n-1}} = 0$, supposing $-p$ and $+q$ to be the Coefficients of the second and third Terms of the proposed Equation.

The 4th Term of any Equation may be taken away by solving a Cubic Equation, which is the Coefficient of the 4th Term in the Equation when transformed, as in the second Article of this Chapter. The 5th Term may be taken away by solving a Biquadratic; and after the same Manner the other Terms can be exterminated if there are any.

503. There

503. There are other Transmutations of Equations that on some Occasions are useful.

An Equation as $x^3 - px^2 + qx - r = 0$, *may be transformed into another that shall have its Roots equal to the Roots of this Equation multiplied by a given Quantity*, as f, by supposing $y = fx$, and consequently $x = \frac{y}{f}$, and substituting this Value for x in the proposed Equation, there will arise $\frac{y^3}{f^3} - \frac{py^2}{f^2} + \frac{qy}{f} - r = 0$, and multiplying all by f^3, we have $y^3 - fpy^2 + f^2 qy - f^3 r = 0$, where the Coefficient of the second Term of the proposed Equation multiplied into f makes the Coefficient of the second Term of the transformed Equation; and the following Coefficients are produced by the following Coefficients of the proposed Equation (as q, r, &c.) multiplied into the Powers of f, (as f^2, f^3, &c.)

Therefore "to transform any Equation into another whose Roots shall be equal to the Roots of the proposed Equation multiplied by a given Quantity" (f), you need only multiply the Terms of the proposed Equation, beginning at the second Term, by f, f^2, f^3, f^4, &c. and putting y instead of x, there will arise an Equation having its Roots equal to the Roots of the proposed Equation, multiplied by (f) as required.

504. The Transformation mentioned in the last Article is of Use when the highest Term of the Equation has a Coefficient different from Unity; for by it, the Equation may be transformed into one that shall have the Coefficient of the highest Term Unit.

If the Equation proposed is $ax^3 - px^2 + qx - r = 0$, then transform the Equation into one whose Roots are equal to the Roots of the proposed Equation multiplied by (a). That is, suppose $y = ax$ or $x = \frac{y}{a}$ and there will arise

$$\frac{ay^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 3; \text{ so that}$$

$$y^3 - py^2 + qay - ra^2 = 0.$$

From which we may draw this

RULE.

RULE.

Change the unknown Quantity x *into another* y, *prefix no Coefficient to the highest Term, pass the second, multiplying the following Terms, beginning with the third, by* a, a^2, a^3, a^4, *&c. the Powers of the Coefficient of the highest Term of the proposed Equation, respectively.*

Thus the Equation $3x^3 - 13x^2 + 14x + 16 = 10$, is transformed into the Equation $y^3 - 13y^2 + 14 \times 3 \times x + 16 \times 9 = 0$, or $y^3 - 13y^2 + 42x + 144 = 0$.

Then finding the Roots of this Equation it will easily be discovered what are the Roots of the proposed Equation: Since $3x = y$, or $x = \frac{1}{3}y$. And therefore since one of the Values of y is -2, it follows that one of the Values of x is $-\frac{2}{3}$.

505. By the last Rule *an Equation is easily cleared of Fractions.* Suppose the Equation proposed is $x^3 - \frac{p}{m}x^2 + \frac{q}{n}x - \frac{r}{c} = 0$. Multiplying all the Terms by the Product of the Denominations, you find

$$mne \times x^3 - nep \times x^2 + meq \times x - mnr = 0.$$

Then, (by last Art.) transforming the Equation into one that shall have Unit for the Coefficient of the highest Term, you find

$$y^3 - nep \times y^2 + m^2e^2nq \times y - m^3n^3e^2r = 0.$$

Or, neglecting the Denomination of the last Term $\frac{r}{e}$, you need only multiply all the Equations by mn, which will give

$$mn \times x^3 - np \times x^2 + mq \times x - \frac{mnn}{e} = 0. \text{ And}$$

then $y^3 - np \times y^2 + m^2nq \times y - \frac{m^3n^3r}{e} = 0$.

Now after the Values of y are found, it will be easy to dis-

cover the Values of x; since, in the first Case, $x = \frac{y}{mne}$; in the second, $x = \frac{y}{mn}$.

For EXAMPLE,

The Equation $x^3 * - \frac{4}{3}x - \frac{146}{27} = 0$, is first reduced to this Form $3x^3 * - 4x - \frac{146}{9} = 0$, and then transformed into $y^3 * - 12y - 146 = 0$.

506. Sometimes, by these Transformations, *Surds are taken away.* As for Example,

The Equation $x^3 - p\sqrt{a} \times x^2 + qx - r\sqrt{a} = 0$, by putting $y = \sqrt{a} \times x$, or $x = \frac{y}{\sqrt{a}}$, is transformed into this Equation, $\frac{y^3}{a\sqrt{a}} - p\sqrt{a} \times \frac{y^2}{a} + q \times \frac{y}{\sqrt{a}} - r\sqrt{a} = 0$. Which by multiplying all the Terms by $a\sqrt{a}$, becomes $y^3 - pay^2 + qay - ra^2 = 0$, an Equation free of Surds. But in order to make this succeed, the Surd ($\sqrt{a}$) must enter the alternate Terms beginning with the second.

507. An Equation, as $x^3 - px^2 + qx - r = 0$, *may be transformed into one whose Roots shall be the Quantities reciprocal of* x; by supposing $y = \frac{1}{x}$, and $y = \frac{z}{r}$, or, (by one Supposition) $x = \frac{r}{z}$, becomes $z^3 - qz^2 + prz - r^2 = 0$.

In the Equation of y, it is manifest that the Order of the Coefficients is inverted; so that if the second Term had been wanting in the proposed Equation, the last but one should have been wanting in the Equations of y and z. If the third had been wanting in the Equation proposed, the last but two had been wanting in the Equations of y and z.

508. Another Use of this Transformation is, that *the greatest Root in the one is transformed into the least Root in the other.* For since $x = \frac{1}{y}$, and $y = \frac{1}{x}$, it is plain that when the Value of x is greatest, the Value of y is least, and conversely.

How

How an Equation is transformed so as to have all its Roots affirmative, shall be explained in the following Chapter.

CHAP. XXIV.

Of finding the Roots of Equations when two or more of the Roots are equal to each other.

509. BEFORE we proceed to explain how to resolve Equations of all Sorts, we shall first demonstrate *how an Equation that has two or more Roots equal, is depressed to a lower Dimension*; and its Resolution made, consequently, more easy. And shall endeavour to explain the Grounds of this and many other Rules we shall give in the remaining Part of this Treatise, in a more simple and concise Manner than has hitherto been done.

In order to this, we must look back to (Inst. 495.) where we find that if any Equation, as $x^3 - px^2 + qx - r = 0$, is proposed, and you are to transform it into another that shall have its Roots less than the Values of x by any given Difference, as e, you are to assume $y = x = e$, and substituting for x its Value $y + e$, you find the transformed Equation,

$$\left.\begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ \quad\; - py^2 - 2pey - pe^2 \\ \qquad\qquad\; + qy \;\; + qe \\ \qquad\qquad\qquad\quad - r \end{array}\right\} = 0.$$

510. Here we are to observe,

1°. That the last Term $(e^3 - pe^2 + qe - r)$ is the very Equation that was proposed, having e in the Place of x.

2°. The Coefficient of the last Term but one is $3e^2 - 2pe + q$, which is the Quantity that arises by multiplying every Term of the last Co-efficient $e^3 - pe^2 + qe - r$ by the Index of e in each Term, and dividing the Product $3e^3 - 2pe^2 + qe$ by the Quantity e that is common to all the Terms.

3°. The

3°. The Coefficient of the laſt Term but two is $3e - p$, which is the Quantity that ariſes by multiplying every Term of the Coefficient laſt found $(3e^2 - 2pe + q)$ by the Index of e in each Term, and dividing the whole by $2e$.

511. Theſe ſame Obſervations extend to Equations of all Dimenſions. If it is the Biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$ that is propoſed, then by ſuppoſing $y = x - e$, it will be transformed into this other,

$$\left.\begin{array}{rrrrr} y^4 + 4ey^3 & + 6e^2y^2 & + 4e^3y & + & e^4 \\ - py^3 & - 3pey^2 & - 3pe^2y & - & pe^3 \\ & + qy^2 & + 2qey & + & qe^2 \\ & & - ry & - & re \\ & & & + & s \end{array}\right\} = 0.$$

Where again it is obvious, that the laſt Term is the Equation that was propoſed, having e in the Place of x. That the laſt Term but one has for its Coefficient the Quantity that ariſes by multiplying the Terms of the laſt Quantity by the Indices of e in each Term, and dividing the Product by e. That the Coefficient of the laſt Term but two, (*viz.* $6e^2 - 3pe + q$) is deduced in the ſame Manner from the Term immediately following, that is, by multiplying every Term of $4e^3 - 3pe^2 + 2qe - r$ by the Index of e in that Term, and dividing the whole by e multiplied into the Index of y in the Term ſought, that is, by $e \times 2$. And the next Term is $4e - p = \dfrac{6e^2 \times 2 - 3pe \times 1}{3e}$.

The Demonſtration of this may eaſily be made general by the Theorem for finding the Powers of a Binomial, ſince the transformed Equation conſiſts of the Powers of the Binomial $y + e$ that are marked by the Indices of e in the laſt Term, multiplied each by their Coefficients $1, -p, +q, -r, +s$, &c. reſpectively.

512. From the laſt two Articles we can eaſily find the Terms of the transformed Equation without any Involution. The laſt Term is had by ſubſtituting e inſtead of x in the propoſed Equation; the next Term, by multiplying every Part of that laſt Term by the Index of e in each Part, and dividing the whole by e; and the following Terms in the Manner deſcribed in the foregoing Article; the reſpective Diviſors being the Quantity e multiplied by the Index of y in each Term.

513. The Demonſtration for finding when two or more Roots are equal, will be eaſy, if we add to this, that *when the unknown Quantity enters all the Terms of any Equation, then one of its Values is equal to nothing*." As in the Equation $x^3 - px^2 + qx = 0$, where $x - 0 = 0$ being one of the ſimple Equations that produce $x^3 - px^2 + qx = 0$, it follows that one of the Values of x is 0. In like Manner two of the Values of x are equal to nothing in this Equation $x^3 - px^2 = 0$; and three of them vaniſh in the Equation $x^4 - px^3 = 0$.

It is alſo obvious (*converſely*) that "if, x does not enter all the Terms of the Equation, *i. e.* if the laſt Term be not wanting, then none of the Values of x can be equal to nothing," for if every Term be not multiplied by x, then $x - 0$ cannot be a Diviſor of the whole Equation, and conſequently 0 cannot be one of the Values of x. If x^2 does not enter into all the Terms of the Equation, then two of the Values of x cannot be equal to nothing. If x^3 does not enter into all the Terms of the Equation, then three of the Values of x cannot be equal to nothing, &c.

514. Suppoſe now that two Values of x are equal to one another, and to e; then it is plain that two Values of y in the transformed Equation will be equal to nothing: Since $y = x - e$. And conſequently, by the laſt Article, the two laſt Terms of the transformed Equation muſt vaniſh.

Suppoſe it is the Cubic Equation of Inſt. 509. that is propoſed, *viz.* $x^3 - px^2 + qx - r = 0$; and becauſe we ſuppoſe $x = e$, therefore the laſt Term of the transformed Equation, *viz.* $e^3 - pe^2 + qe - r$ will vaniſh.* And ſince two Values of y vaniſh, the laſt Term but one, *viz.* $3e^2y - 2pey + qy$ will vaniſh at the ſame Time. So that $3e^2 - 2pe + q = 0$. But, by Suppoſition, $e = x$; therefore, when two Values of x, in the Equation $x^3 - px^2 + qx - r = 0$, are equal, it follows, that $3x^2 - 2px + q = 0$. And thus "the propoſed Cubic is depreſſed to a Quadratic that has one of its Roots equal to one of the Roots of that Cubic."

If

* Becauſe ſince $x = e$, therefore $y = x - e = 0$, and conſequently all the Terms in which y is found will vaniſh; and they are all but the laſt, therefore the laſt Term $e^3 - pe^2 + qe - r$ will be left equal to nothing, ſince the whole Equation was ſo at firſt.

If it is the Biquadratic that is propoſed, *viz.* $x^4 - px^3 + qx^2 - rx + s = 0$, and two of its Roots be equal; then ſuppoſing $e = x$, two of the Values of y muſt vaniſh, and the Equation of Inſt. 511. will be reduced to this Form,

$$\left.\begin{array}{l} y^4 + 4ey^3 + 6e^2y^2 \\ \quad\; - py^3 - 3pey^2 \\ \qquad\qquad\; + qy^2 \end{array}\right\} * * = 0. \text{ So that}$$

$4e^3 - 3pe^2 + 2qe - r = 0$; or becauſe $x = e$,
$4x^3 - 3px^2 + 2qx - r = 0$.

515. In general, when two Values of x are equal to each other, and to e, the two laſt Terms of the transformed Equation vaniſh: And conſequently, "if you multiply the Terms of the propoſed Equation by the Indices of x in each Term, the Quantity that will ariſe will be $= 0$, and will give an Equation of a lower Dimenſion than the propoſed, that ſhall have one of its Roots equal to one of the Roots of the propoſed Equation."

That the two laſt Terms of the Equation vaniſh when the Values of x are ſuppoſed equal to each other, and to e, will alſo appear by conſidering, that ſince two Values of y then become equal to nothing, the Product of the Values of y muſt vaniſh, which is equal to the laſt Term of the Equation; and becauſe two of the four Values of y are equal to nothing, it follows alſo, that one of any three that can be taken out of theſe four muſt be $= 0$; and therefore, the Products made by multiplying any three muſt vaniſh; and conſequently the Coefficient of the laſt Term but one, which is equal to the Sum of theſe Products, muſt vaniſh.

516. After the ſame Manner, if there are three equal Roots in the Biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$, and if e be equal to one of them; three Values of y $(= x - e)$ will vaniſh, and conſequently y^3 will enter all the Terms of the transformed Equation; which will have this Form,

$$\left.\begin{array}{l} y^4 + 4ey^3 \\ \quad\; - py^3 \end{array}\right\} * * * = 0. \text{ So that here}$$

$6e^2 - 3pe + q = 0$; or, ſince $e = x$, therefore,
$6x^2 - 3px + q = 0$: And one of the Roots of this Quadratic will be equal to one of the Roots of the propoſed Biquadratic.

 In

In this Caſe, two of the Roots of the Cubic Equation $4x^3 - 3px^2 + 2qx - r = 0$ are Roots of the propoſed Biquadratic, becauſe the Quantity $6x^2 - 3px + q$ is deduced from $4x^3 - 3px^2 + 2qx - r$, by multiplying the Terms by the Indexes of x in each Term.

In general, " whatever is the Number of equal Roots in the propoſed Equation, they will all remain but one in the Equation that is deduced from it, by multiplying all the Terms by the Indexes of x in them; and they will all remain but two in the Equation deduced in the ſame Manner from That;" and ſo of the reſt.

517. What we obſerved of the Coefficients of Equations transformed by ſuppoſing $y = x - e$, leads to this eaſy Demonſtration of this Rule; and will be applied, in the next Chapter, to demonſtrate the Rules for finding the Limits of Equations.

It is obvious however, that tho' we make Uſe of Equations whoſe Signs change alternately, the ſame Reaſoning extends to all other Equations.

518. It is a Conſequence alſo of what has been demonſtrated, that " if two Roots of any Equation, as,

$$x^3 - px^2 + qx - r = 0, \text{ are equal,}$$

then multiplying the Terms by any arithmetical Series, as,

$$a + 3b,\ a + 2b,\ a + b,\ a, \text{ the Product will be} = 0."$$

For ſince

$$ax^3 - apx^2 + aqx - ar = 0; \text{ and}$$

$$\overline{3x^2 - 2px + q} \times bx = 0, \text{ it follows, that}$$

$$ax^3 + 3bx^3 - apx^2 - 2bpx^2 + aqx + bqx - ar = 0.$$

Which is the Product that ariſes by multiplying the Terms of the propoſed Equation by the Terms of the Series, $a + 3b$, $a + 2b$, $a + b$, a; which may repreſent any Arithmetical Progreſſion.

CHAP.

CHAP. XXV.

Of the Limits of Equations.

519. WE now proceed to ſhew how to diſcover the Limits of the Roots of Equations, by which their Solution is much facilitated.

Let any Equation, as $x^3 - px^2 + qx - r = 0$ be propoſed; and transform it, as above, into the Equation

$$\left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array}\right\} = 0.$$

Where the Values of y are leſs than the reſpective Values of x by the Difference e. If you ſuppoſe e to be taken ſuch as to make all the Coefficients, of the Equation of y, poſitive, *viz.* $e^3 - pe^2 + qe - r$, $3e^2 - 2pe + q$, $3e - p$; then there being no Variation of the Signs in the Equation, all the Values of y muſt be negative; and conſequently, the Quantity e, by which the Values of x are diminiſhed, muſt be greater than the greateſt poſitive Value of x: And conſequently muſt be the Limit of the Roots of the Equation $x^3 - px^2 + qx - r = 0$.

It is ſufficient therefore, in order to find the Limit, to "enquire what Quantity ſubſtituted for x in each of theſe Expreſſions $x^3 - px^2 + qx - r$, $3x^2 - 2px + q$, $3x - p$, will give them all poſitive;" for that Quantity will be the Limit required.

How theſe Expreſſions are formed from one another, was explained in the Beginning of the laſt Chapter.

EXAMPLE.

520. If the Equation $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x + 120 = 0$ is propoſed; and it is required to determine the Limit that is greater than any of the Roots; you are to enquire what integer Number ſubſtituted for x in the propoſed Equation, and following Equations deduced from it by Inſt. 512, will give, in each, a poſitive Quantity.

$$5x^4 - 8x^3 - 30x^2 + 60x + 63$$
$$5x^3 - 6x^2 - 15x + 15$$
$$5x^2 - 4x - 5$$
$$5x - 2.$$

The least integer Number which gives each of these positive is 2; which therefore is the Limit of the Roots of the proposed Equation; or a Number that exceeds the greatest positive Root.

521. If the Limit of the *negative* Roots is required, you may (by Inst. 494.) change the negative into positive Roots, and then proceed as before to find their Limits. Thus, in the Example, you will find that -3 is the Limit of the negative Roots. So that the five Roots of the proposed Equation are betwixt -3 and $+2$.

522. Having found the Limit that surpasses the greatest positive Root, call it m. And if you assume $y = m - x$, and for x substitute $m - y$, the Equation that will arise will have all its Roots positive; because m is supposed to surpass all the Values of x, and consequently $m - x$ $(= y)$ must always be affirmative. And by this Means, *any Equation may be changed into one that shall have all its Roots affirmative.*

Or if $-n$ represent the Limit of the negative Roots, then by assuming $y = x + n$, the proposed Equation shall be transformed into one that shall have all its Roots affirmative; for $+n$ being greater than any negative Value of x, it follows, that $y = x + n$ must be always positive.

523. *The greatest negative Coefficient of any Equation increased by Unit, always exceeds the greatest Root of the Equation.*

To demonstrate this, let the Cubic $x^3 - px^2 - qx - r = 0$ be proposed; where all the Terms are negative except the first. Assuming $y = x - e$ it will be transformed into the following Equation,

$$(A)\quad \left.\begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ \quad\;\; - py^2 - 2pey - pe^2 \\ \qquad\qquad\;\; - qy - qe \\ \qquad\qquad\qquad\;\; - r \end{array}\right\} = 0.$$

1°. Let us suppose that the Coefficients p, q, r, are equal to each other; and if you also suppose $e = p + 1$, then the last Equation becomes

(B) $y^3 +$

$$(B)\ \left.\begin{array}{l} y^3 + 2py^2 + p^2 y + 1 \\ \quad + 3y^2 + 3py \\ \qquad\quad + 3y \end{array}\right\} = 0.$$

Where all the Terms being poſitive, it follows, that the Values of y are all negative, and that conſequently e, or $p + 1$, is greater than the greateſt Value of x in the propoſed Equation.

2°. If q and r be not $= p$, but leſs than it, and for e you ſtill ſubſtitute $p + 1$ (ſince the negative Part $\left(\begin{array}{l} -qy - qe \\ \qquad\ -r \end{array}\right)$ becomes leſs, the poſitive remaining undiminiſhed,) all the Coefficients of the Equation (A) become poſitive. And the ſame is obvious if q and r have poſitive Signs, and not negative Signs, as we ſuppoſed. It appears therefore, "that, if, in any Cubic Equation, p be the greateſt negative Coefficient, then $p + 1$ muſt ſurpaſs the greateſt Value of x."

524. 3°. By the ſame Reaſoning it appears, that if q be the greateſt negative Coefficient of the Equation, and $e = q + 1$, then there will be no Variation of the Signs in the Equation of y: For it appears from the laſt Article, that if all the three (p, q, r,) were equal to one another, and e equal to any one of them increaſed by Unit, as to $q + 1$, then all the Terms of the Equation (A) would be poſitive. Now if e be ſuppoſed ſtill equal to $q + 1$, and p and r to be leſs than q, then all theſe Terms will be poſitive, the negative Part, which involves p and r being diminiſhed, while the poſitive Part and the negative involving q remain as before.

4°. After the ſame Manner it is demonſtrated that if r is the greateſt negative Coefficient in the Equation, and e is ſuppoſed $= r + 1$, then all the Terms of the Equation (A) of y will be poſitive; and conſequently $r + 1$ will be greater than any of the Values of x.

What we have ſaid of the *Cubic* Equation $x^3 - px^2 + qx - r = 0$, is eaſily applicable to others.

525. *In general*, we conclude, that "the greateſt negative Coefficient in any Equation increaſed by Unit, is always a Limit that exceeds all the Roots of that Equation."

But it is to be obſerved at the ſame Time, that the greateſt negative Coefficient increaſed by Unit, is very ſeldom the *neareſt* Limit: That is beſt diſcovered by the Rule in Inſt. 519.

526. As

526. As our Plan obliges us to take in every uſeful Part of Science, but no more, we ſhall here leave the remaining Part of Mr. *Maclaurins*'s Diſcourſe on the *Limits of Equation*, and proceed to the ſeveral Methods of *ſolving Equations*; of theſe we ſhall propoſe ſeveral from different Authors, ſome being preferable in one Reſpect, and ſome in another, and the Learner will be thus enabled to chuſe that which will beſt anſwer his Purpoſe in any Caſe propoſed.

527. We ſhall begin again upon this Subject with Mr. *Maclaurin*; but becauſe his Method of ſolving Equations requires the Learner to have a juſt Idea of *Commenſurable Quantities* (for his Method extends only to ſuch Equations whoſe Roots are *commenſurable*,) therefore we muſt premiſe the following Propoſition, *viz.*

528. *All Commenſurable Quantities are to each other as Number to Number, that is, their Proportion may be expreſſed by whole Numbers*, or they *have one common Number that will meaſure them all.*

Thus for Inſtance, let a and b be two Commenſurable Quantities, then their Proportion, be it what it will, may be expreſſed in *whole Numbers*; for let c be their common Meaſure without Remainder, and let it meaſure a juſt three Times, and b four Times; then will $3c = a$, and $4c = b$; therefore $a : b :: (3c : 4c ::)\ 3 : 4$ *Q.E.D.*

529. Hence (*vice verſa*) all Quantities that are to one another as Number to Number, are *Commenſurable*. Hence alſo all *whole Numbers* are commenſurable, ſince Unity is a common Meaſure to them all.

530. Likewiſe all *Fractions* are *commenſurable*, as $\frac{a}{b}$ and $\frac{c}{d}$; for if they are reduced to a common Denomination, they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$, and $\frac{1}{bd}$ will meaſure them both.

531. *Incommenſurable Quantities* are ſuch whoſe Proportion cannot be expreſſed in whole Numbers or finite Fractions, or they can have no common Meaſure, but their Relation muſt be expreſſed by a *Surd Quantity* or *infinite Series*. Thus the Ratio of 1 to $\sqrt{2}$, cannot be expreſſed in Numbers, nor the Diameter (d) to the Circumference of a Circle (c) without an infinite Series, as we ſhall hereafter ſee.

CHAP.

CHAP. XXVI.

Of the Resolution of Equations, all whose Roots are commensurate.

532. IT was demonstrated, in Inst. 482. that the last Term of any Equation is the Product of its Roots: From which it follows, that the Roots of an Equation, when commensurable Quantities, will be found among the Divisors of the last Term. And hence we have for the Resolution of Equations, this

RULE.

Bring all the Terms to one Side of the Equation, find all the Divisors of the last Term, and substitute them successively for the unknown Quantity in the Equation. So shall that Divisor which, substituted in this Manner, gives the Result $=0$, *be the Root of the proposed Equation.*

For Example, suppose this Equation is to be resolved,

$$\left.\begin{array}{l} x^3 - 3ax^2 + 2a^2x - 2a^2b \\ \quad - bx^2 + 3abx \end{array}\right\} = 0,$$

where the last Term is $2a^2b$, whose simple literal Divisors are a, b, $2a$, $2b$, each of which may be taken either positively or negatively. But as here we find there are Variations of Signs in the Equation, we need only take them positively. Suppose $x = a$ the first of the Divisors, and substituting a for x, the Equation becomes

$$\left.\begin{array}{l} a^3 - 3a^3 + 2a^3 - 2a^2b \\ \quad - a^2b + 3a^2b \end{array}\right\} \text{or, } 3a^3 - 3a^3 + 3a^2b - 3a^2b = 0.$$

So that the whole vanishing, it follows that a is one of the Roots of the Equation.

533. After the same Manner, if you substitute b in the Place of x, the Equation is,

$$\left.\begin{array}{l} b^3 - 3ab^2 + 2a^2b - 2a^2b \\ -b^3 + 3ab^2 \end{array}\right\} = 0,$$

which vanishing shews b to be another Root of the Equation.

Again, if you substitute $2a$ for x, you will find all the Terms destroy one another so as to make the Sum $= 0$. For it will then be,

$$\left.\begin{array}{l} 8a^3 - 12a^3 + 4a^3 - 2a^2b \\ \quad - 4a^2b + 6a^2b \end{array}\right\} = 0.$$

Whence we find, that $2a$ is the third Root of the Equation. Which, after the first two ($+a$, $+b$,) had been found, might have been collected from this, that the last Term being the Product of the three Roots, $+a$, $+b$ being known, the third Term must necessarily be equal to the last Term divided by the Product ab, that is, $= \frac{2a^2b}{ab} = 2a$.

534. Let the Roots of the Cubic Equation

$x^3 - 2x^2 - 33x + 90 = 0$ be required.

And first the Divisors of 90 are found to be 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90. If you substitute 1 for x, you will find $x^3 - 2x^2 - 33x + 90 = 56$; so that one is not a Root of the Equation. If you substitute 2 for x, the Result will be 24: but, putting $x = 3$, you have

$$x^3 - 2x^2 - 33x + 90 = 27 - 18 - 99 + 90 = 117 - 117 = 0.$$

So that 3 is one of the Roots of the proposed Equation. The other affirmative Root is $+5$; and after you find it, as it is manifest from the Equation, that the other Root is negative, you are not to try any more Divisors taken positively, but to substitute them, negatively taken, for x: And thus you find that -6 is the third Root. For putting $x = -6$, you have

$$x^3 - 2x^2 - 33x + 90 = -216 - 72 + 198 + 90 = 0.$$

This last Root might have been found by dividing the last Term 90, having its Sign changed, by 15, the Product of the two Roots already found.

535. When one of the Roots of an Equation is found, in order to find the rest with less Trouble, divide the proposed Equation

tion by the simple Equation which you are to deduce from the Root already found, and the Quotient shall give an Equation of a Degree lower than the proposed; whose Roots will give the remaining Roots required.

As for Example, the Root $+3$, first found, gave $x = 3$ or $x - 3 = 0$, whence dividing thus,

$$\begin{array}{l} x-3)\ x^3 - 2x^2 - 33x + 90\ (x^2 + x - 30 \\ \quad\quad\ \ x^3 - 3x^2 \\ \hline \quad\quad\quad\quad\ \ x^2 - 33x + 90 \\ \quad\quad\quad\quad\ \ x^2 - 3x \\ \hline \quad\quad\quad\quad\quad - 30x + 90 \\ \quad\quad\quad\quad\quad - 30x + 90 \\ \hline \quad\quad\quad\quad\quad\quad 0 \quad\quad 0 \end{array}$$

The Quotient shall give a Quadratic Equation $x^2 + x - 30 = 0$, which must be the Product of the other two simple Equations from which the Cubic is generated, and whose Roots therefore must be two of the Roots of that Cubic.

Now the Roots of that Quadratic Equation are easily found (by Inst. 339.) to be $+5$ and -6. For,

$$x^2 + x = 30$$

add $\frac{1}{4}$ . . $x^2 + x + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4}$

$\surd$ . . . $x + \frac{1}{2} = \pm\sqrt{\frac{121}{4}} = \pm\frac{11}{2}$

and . . . $x = \pm\frac{11}{2} - \frac{1}{2} = +5$ or -6.

536. After the same Manner, if the Biquadratic $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$ is to be resolved; by substituting the Divisors of 120 for x, you will find that $+3$, one of those Divisors is one of the Roots; the Substitution of 3 for x giving $81 - 54 - 225 + 78 + 120 = 279 - 279 = 0$. And therefore dividing the proposed Equation by $x - 3$, you must enquire for the Roots of the Cubic $x^3 + x^2 - 22x - 40 = 0$, and finding that $+5$, one of the Divisors of 40, is one of the Roots, you divide that Cubic by $x - 5$, and the Quotient gives the Quadratic $x^2 + 6x + 8 = 0$, whose two Roots are -2, -4. So that the four Roots of the Biquadratic are $+3$, $+5$, -2, -4.

 537. This

537. This Rule supposes that you can find all the Divisors of the last Term; which you may always do thus.

If it is a simple Quantity, divide it by its least Divisor that exceeds Unit, and the Quotient again by its least Divisor, proceeding thus till you have a Quotient that is not divisible by any Number greater than Unit. This Quotient, with these Divisors, are the first or *simple Divisors* of the Quantity. And the Products of the Multiplication of any 2, 3, 4, &c. of them are the *compound Divisors.*

As, to find the Divisors of 60; first I divide by 2, and the Quotient 30 again by 2, then the next Quotient 15 by 3, and the Quotient of this Division 5 is not farther divisible by any Integer above Units; so that the simple Divisors are,

2, 2, 3, 5;
The Products of two, 4, 6, 10, 15.
The Products of three, 12, 20, 30.
The Product of all four, ... 60.

The Divisors of 90 are found after the same Manner;

Simple Divisors, 2, 3, 3, 5.
The Products of two, 6, 9, 10, 15.
The Products of three, 18, 30, 45.
The Product of all four, ... 90.

The Divisors of $21abb$.

The simple Divisors, 3, 7, a, b, b.
The Products of two, 21, $3a$, $3b$, $7a$, $7b$, ab, bb.
The Products of three, $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb.
The Products of four, $21ab$, $21bb$, $3abb$, $7abb$.
The Products of the five, ... $21abb$.

538. But as the last Term may have very many Divisors, and the Labour may be very great to substitute them all for the unknown Quantity, we shall now shew how it may be abridged, by limiting to a small Number the Divisors you are to try. And, first it is plain, (from Inst. 523.) that "any Divisor that exceeds the greatest negative Coefficient by Unity is to be neglected."

ed." Thus in resolving the Equation $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$, as 25 is the greatest negative Coefficient, we conclude that the Divisors of 120 that exceed 26 may be neglected.

539. But the Labour may be still abridged, if we make Use of the Rule in Inst. 519. that is, if we find the Number which substituted in these following Expressions,

$$x^4 - 2x^3 - 25x^2 + 26x + 120,$$
$$2x^3 - 3x^2 - 25x + 13,$$
$$6x^2 - 6x - 25$$
$$2x - 1,$$

will give in them all a positive Result: For that Number will be greater than the greatest Root, and all the Divisors of 120 that exceed it may be neglected.

That this Investigation may be easier, we ought to begin always with that Expression, where the negative Roots seem to prevail most; as here in the Quadratic Expression $6x^2 - 6x - 25$; where finding that 6 substituted for x gives that Expression positive, and gives all the other Expressions at the same Time positive, I conclude that 6 is greater than any of the Roots, and that all the Divisors of 120 that exceed 6 may be neglected.

540. If the Equation $x^3 + 11x^2 + 10x - 72 = 0$ is proposed, the Rule of Inst. 523. does not help to abridge the Operation; the last Term itself being the greatest negative Term. But, by Inst. 519. we enquire what Number substituted for x will give all these Expressions positive.

$$x^3 + 11x^2 + 10x - 72$$
$$3x^2 + 22x + 10$$
$$3x + 11$$

Where the Labour is very short, since we need only attend to the first Expression; and we see immediately that 4 substituted for x gives a positive Result, whence all the Divisors of 72 that exceed 4 are to be rejected; and thus, by a few Trials, we find that $+ 2$ is the positive Root of the Equation. Then dividing the Equation by $x - 2$, and resolving the Quadratic Equation that is the Quotient of the Division, you find the other two Roots to be $- 9$, and $- 4$.

541. Besides

541. Besides the Method already explained, there are others by which Limits may be determined, which the Root of an Equation cannot exceed.

Since the Squares of all real Quantities are affirmative, it follows, that *The Sum of the Squares of the Roots of any Equation must be greater than the Square of the greatest Root.* And the Square Root of that Sum will therefore be a Limit that must exceed the greatest Root of the Equation.

If the Equation proposed is $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$, &c. $= 0$, then the Sum of the Squares of the Roots (by Inst. 485.) will be $p^2 - 2q$. So that $\sqrt[2]{p^2 - 2q}$ will exceed the greatest Root of that Equation.

Or if you find, (by Inst. 486.) the Sum of the 4th Powers of the Roots of the Equation, and extract the Biquadratic Root of that Sum, it will also exceed the greatest Root of the Equation.

542. But there is another Method that reduces the Divisors of the last Term, that can be useful, still to more narrow Limits.

Suppose the Cubic Equation $x^3 - px^2 + qx - r = 0$ is proposed to be resolved. Transform it to an Equation whose Roots shall be less than the Values of x by Unity, assuming $y = x - 1$. And the last Term of the transformed Equation will be $1 - p + q - r$; which is found by substituting Unit, the Difference of x and y, for x, in the proposed Equation; as will easily appear from Inst. 495. where, when $y = x - e$, the last Term of the transformed Equation was $e^3 - pe^2 + qe - r$.

Transform again the Equation $x^3 - px^2 + qx - r = 0$, by assuming $y = x + 1$, into an Equation whose Roots shall exceed the Values of x by Unit, and the last Term of the transformed Equation will be $-1 - p - q - r$, the same that arises by substituting -1, the Difference betwixt x and y, for x in the proposed Equation.

Now the Values of x are some of the Divisors of r, which is the Term left when you suppose $x = 0$; and the Values of the y's are some of the Divisors of $+1 - p + q - r$, and of $-1 - p - q - r$, respectively. And these Values are in Arithmetical Progression increasing by the common Difference Unit; because $x - 1$, x, $x + 1$, are in that Progression. And it is obvious

obvious the same Reasoning may be extended to any Equation of whatever Degree. So that this gives a general Method for the Resolution of Equations whose Roots are commensurable.

RULE.

Substitute in Place of the unknown Quantity successively the Terms of the Progression 1, 0, — 1, &c. *and find all the Divisors of the Sums that result; then take out all the arithmetical Progressions you can find among these Divisors, whose common Difference is Unit; and the Values of* x *will be among the Divisors arising from the Substitutions of* $x = 0$ *that belong to these Progressions.* The Values of x will be affirmative when the arithmetical Progression increases, but negative when it decreases.

EXAMPLE.

543. Let it be required to find one of the Roots of the Equation $x^3 - x^2 - 10x + 6 = 0$. The Operation is thus;

Supposit.	*Result.*		*Divisors.*	*Arith. Prog. decr.*	
$x = 1$	. . . .	-4	1,2,4	4	
$x = 0$	$x^3 - x^2 - 10x + 6 =$	$+6$	1,2,3, 6	3	gives $x = -3$
$x = -1$	. . . .	$+14$	1,2,7,14	2	

Where the Suppositions of $x = 1$, $x = 0$, $x = -1$ give the Quantity $x^3 - x^2 - 10x + 6$ equal to -4, 6, 14; among whose Divisors we find only one arithmetical Progression 4, 3, 2; the Term of which opposite to the Supposition of $x = 0$, being 3, and the Series decreasing, we try if -3 substituted for x makes the Equation vanish; which succeeding one of its Roots must be -3. Then dividing the Equation by $x + 3$, we find the Roots of the (Quadratic) Quotient $x^2 - 4x + 2 = 0$ are $2 \pm \sqrt{2}$.

544. If it is required to find the Roots of the Equation $x^3 - 3x^2 - 46x - 72 = 0$, the Operation will be thus;

Suppos.	*Results*	*Divisors.*	*Progressions.*
$x = 1$	-120	1,2,3,4,5,6,8,10,12,15,20,24,30,40,60,120.	8 3 4 5
$x = 0$	-72	1,2,3,4,6,8,9,12,18,24,36,72.	9 2 3 4
$x = -1$	-30	1,2,3,5,6,10,15,30.	10 1 2 3

Of these four Arithmetical Progressions having their common Difference equal to Unit, the first gives $x = 9$, the others give $x = -2$, $x = -3$, $x = -4$; all which succeed except $x = -3$: So that the three Values of x are $+9$, -2, -4.

CHAP.

CHAP. XXVII.

Of the Resolution of Equations by Cardan's *Rule,*† *and others of that Kind.*

545. THE second Term can be taken away out of any *Cubic* Equation, by Inst. 497; so that they all may be reduced to this Form,

$$x^3 * + qx + r = 0.$$

Let us suppose that $x = a + b$; and $x^3 + qx + r = a^3 + 3a^2b + 3ab^2 + b^3 + qx + r = a^3 + 3ab \times \overline{a+b} + b^3 + qx + r = a^3 + 3abx + b^3 + qx + r =$ (by supposing $3ab = -q$) $= a^3 + b^3 + r = 0$.

But $b = -\frac{q}{3a}$, and $b^3 = -\frac{q^3}{27a^3}$ and consequently, $a^3 - \frac{q^3}{27a^3} + r = 0$; or, $a^6 + ra^3 = \frac{q^3}{27}$.

Suppose $a^3 = z$, and you have, $z^2 + rz = \frac{q^3}{27}$; which is a Quadratic whose Resolution gives

$$z = -\tfrac{1}{2}r \pm \sqrt{\tfrac{1}{4}r^2 + \frac{q^3}{27}} = a^3,$$

$$\text{and } a = \sqrt[3]{-\tfrac{1}{2}r \pm \sqrt[2]{\tfrac{1}{4}r^2 + \frac{q^3}{27}}}; \text{ and}$$

$$x = a + b = a - \frac{q}{3a} = \sqrt[3]{-\tfrac{1}{2}r \pm \sqrt[2]{\tfrac{1}{4}r^2 + \frac{q^3}{27}}} - \frac{q}{3 \times \sqrt[3]{-\tfrac{1}{2}r \pm \sqrt[2]{\tfrac{1}{4}r + \frac{q^3}{27}}}}$$

: in which Expressions there are only known Quantities.

 546. The

† Tho' this Method be generally ascribed to *Cardan,* yet *Cardan* himself mentions it as the Invention of one *Scipio Ferreus,* a noted Mathematician before his Time.

546. The Values of x may be found a little differently, thus;

Since $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, it follows,

that $a^3 + r = + \frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, and

$b^3 (= -a^3 - r) = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$; so that $b =$

$\sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}$; and $x\ (= a + b) =$

$$\sqrt[3]{-\frac{1}{2}r \pm \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}};$$

which gives but one Value of x, because when, in the Value of a the Surd $\sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ is positive, it is negative in the Value of b, and there is only the Difference of this Sign in their Values. So that we may conclude

$$x = \sqrt[3]{-\frac{1}{2}r + \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}.$$

547. * The Values of x may be discovered without exterminating the second Term.

Any Cubic Equation may be reduced to this Form,

$$\left.\begin{array}{r} x^3 - 3px^2 - 3qx - 2r \\ + 3p^2x - p^3 \\ + 3pq \end{array}\right\} = 0,$$

which by supposing $x = z + p$, will be reduced to $z^3 * - 3qz - 2r = 0$, in which the second Term is wanting. But by the last Article, since $z^3 * - 3qz - 2r = 0$, it follows that

$z =$

* This Method we owe to the Reverend and very Learned Mr. John Colson, the present *Lucasian* Professor of Mathematics in the University of Cambridge, and Successor to Dr. *Saunderson*. See Philos. Trans. 309.

$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}} =$ (if you suppose that the Cubic Root of the Binomial $r + \sqrt{r^2 - q^3}$ is $m + \sqrt{n}$) $= m + \sqrt{n} + m - \sqrt{n} = 2m$. And since $x = z + p$, it follows that $x = p + 2m$.

548. But as the square Root of any Quantity is *two-fold*, "the Cube Root is *three-fold*," and can be expressed three different Ways.

Suppose the Cube Root of Unit is required, and let $y^3 = 1$, or $y^3 - 1 = 0$, then since Unit itself is a Cube Root of 1, one of the Values of y is 1, so that the Equation $y - 1 = 0$ shall divide the first Equation $y^3 - 1 = 0$, and the Quotient $y^2 + y + 1 = 0$, resolved, gives $y = \frac{-1 \mp \sqrt{-3}}{2}$; so that the 3 Expressions of $\sqrt[3]{1}$ are 1, $\frac{-1 + \sqrt{-3}}{2}$, and $\frac{-1 - \sqrt{-3}}{2}$.

And, in general, the Cube Root of any Quantity A^3 † may be A, or $\frac{-1 + \sqrt{-3}}{2} \times A$, or $\frac{-1 - \sqrt{-3}}{2} \times A$; so that the Cube Root of the Binomial $r + \sqrt{r^2 - q^3}$ may be $m + \sqrt[2]{n}$, as we supposed above, or $\frac{-1 + \sqrt{-3}}{2} \times \overline{m + \sqrt[2]{n}}$, or $\frac{-1 - \sqrt{-3}}{2} \times \overline{m + \sqrt[2]{n}}$. And hence we have three Expressions for x, *viz.*

1. $x = p + 2m$,
2. $x = p - m + \sqrt{-3n}$.
3. $x = p - m - \sqrt{-3n}$.

and these give the three Roots of the proposed Cubic Equation.

EXAMPLE I.

549. Let it be required to find the Roots of the Equation $x^3 - 12x^2 + 41x - 42 = 0$.

† Since $A^3 = A^3 \times 1$, and $\sqrt[3]{A^3} \times \sqrt[3]{1} = A$.

Comparing the Coefficients of this Equation with those of the general Equation

$$x^3 - 3p\,x^2 \left.\begin{matrix} -3q \\ +3p^2 \end{matrix}\right\} x \left.\begin{matrix} -2r \\ -p^3 \\ +3pq \end{matrix}\right\} = 0, \text{ you find,}$$

1. $3p = 12$, so that . . . . $p = 4$,
2. $3p^2 - 3q\,(= 48 - 3q) = 41$ . . . . . . $q = \frac{7}{3}$,
3. $3pq - p^3 - 2r\,(= -36 - 2r) = -42$. $r = 3$;

and consequently, $r^2 - q^3 = 9 - \frac{343}{27} = -\frac{100}{27}$, and $r + \sqrt{r^2 - q^3} = 3 + \sqrt{-\frac{100}{27}}$. Now the Cube Root of this Binomial is found to be, $-1 + \sqrt[2]{-\frac{4}{3}}$ $(= m + \sqrt{n}.)$† Whence

1. $x = p + 2m = 4 - 2 = 2$.
2. $x = p - m - \sqrt{-3n} = 4 + 1 - \sqrt{4} = 5 - 2 = 3$.
3. $x = p - m + \sqrt{-3n} = 5 + 2 = 7$.

So that the three Roots of the proposed Equation are 2, 3, 7.

You may find other two Expressions of the Cube Root of $3 + \sqrt{-\frac{100}{27}}$, besides $-1 + \sqrt{-\frac{4}{3}}$, *viz.* $\frac{3}{2} + \sqrt{-\frac{1}{12}}$, and $-\frac{1}{2} - \sqrt{-\frac{25}{12}}$; but these substituted for $m + \sqrt{n}$ give the same Values for x as are already found.

EXAMPLE II.

550. In the Equation $x^3 + 15x^2 + 84x - 100 = 0$, you find $p = -5$, $q = -3$, $r = 135$, and $r + \sqrt{r^2 - q^3} = 135 + \sqrt{18252}$, whose Cube Root is $3 + \sqrt{12}$; so that $x = (p + 2m) = -5 + 6 = 1$. The other two Values of x, *viz.* $-8 + \sqrt{-36}$, $-8 - \sqrt{-36}$, are impossible. For in this Case $n = 12$, therefore $-3n = -36$, which makes the Quantity $\sqrt{-3n}$, and of Course the two last Roots, impossible.

After the same Manner you will find that the Roots of the Equation $x^3 + x^2 - 166x + 660 = 0$, are -15, $7 \pm \sqrt{5}$.

And

† We shall, hereafter, shew, from Mr. *De Moivre*, how the Cubic Root of an impossible Binomial, as $a + \sqrt{-b}$, may be extracted by the *Trisection* of an Angle.

And here we are to observe, that whenever this irrational Part, $\sqrt{r^2 - q^3}$, is impossible, that is, as often as q is an affirmative Quantity, and at the same Time its Cube is greater than r^2, the three Roots of the Equation are all possible and real; but if $\sqrt{r^2 - q^3}$ be possible, that is, if q be negative, or if affirmative, its Cube be less than r^2, the Equation will have but one possible and real Root, the other two being impossible or imaginary. This is easily deduced from the Structure of the Equation, as we have just now shewn.

Note, If in this general Theorem we put $p = 0$, the second Term vanishes, and the Equation is reduced to *Cardan*'s Forms before specified.

551. The Roots of Biquadratic Equations may be found by reducing them to Cubes, thus.

Let the second Term be taken away by the Rule given in Inst. . and let the Equation that results be,

$$x^4 * + qx^2 + rx + s = 0.$$

And let us suppose this Biquadratic to be the Product of these two Quadratic Equations,

$$\begin{array}{l} x^2 + ex + f = 0 \\ x^2 - ex + g = 0 \\ \hline x^4 * \left.\begin{array}{l} +f \\ +g \\ -e^2 \end{array}\right\} \times x^2 \left.\begin{array}{l} +eg \\ -ef \end{array}\right\} \times x + fg = 0. \end{array}$$

Where e is the Coefficient of x in both Equations but effected with contrary Signs; because when the second Term is wanting in an Equation, the Sum of the affirmative Roots must be equal to the Sum of the negative.

Compare now the proposed Equation with the above Product, and the respective Terms put equal to each other will give $f + g - e^2 = q$, $eg - ef = r$, $fg = s$. Whence it follows that $f + g = q + e^2$, and $g - f = \frac{r}{e}$; and consequently,

$$f + g + g - f \; (= 2g) = q + e^2 + \frac{r}{e}, \text{ and}$$

$g =$

$g = \dfrac{q + e^2 + \frac{r}{e}}{2}$; the same Way, you will find, by Subtraction, &c. $f = \dfrac{q + e^2 - \frac{r}{e}}{2}$, and $f \times g\ (= s) = \frac{1}{4} \times \overline{q^2 + 2qe^2 + e^4 - \frac{r^2}{e^2}}$; and multiplying by $4e^2$ and ranging the Terms, you have this Equation,

$$e^6 + 2qe^4 + \overline{q^2 - 4s} \times e^2 - r^2 = 0.$$

Suppose $e^2 = y$, and it becomes $y^3 + 2qy^2 + \overline{q^2 - 4s}\, y - r^2 = 0$, a Cubic Equation whose Roots are to be discovered by the preceding Articles. Then the Values of y being found, their Square Roots will give e (since $y = e^2$;) and having e, you will find f and g from the Equations $f = \dfrac{q + e^2 - \frac{r}{e}}{2}$, $g = \dfrac{q + e^2 + \frac{r}{e}}{2}$. Lastly extracting the Roots of the Equations $x^2 + ex + f = 0$, $x^2 - ex + g = 0$, you will find the four Roots of the Biquadratic $x^4 * qx^2 + rx + s = 0$; for either $x = -\frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - f}$, or, $x = +\frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - g}$.

552. Or if you want to find the Roots of the Biquadratic without taking away the second Term;

Suppose it to be of this Form,

$$x^4 - 4px^3 \left.\begin{matrix} -2q \\ +4p^2 \end{matrix}\right\} x^2 \left.\begin{matrix} -8r \\ +4pq \end{matrix}\right\} x \left.\begin{matrix} -4s \\ +q^2 \end{matrix}\right\} = 0,$$

and the Values of x will be

$$\left.\begin{matrix} x = p - a \pm \sqrt{p^2 + q - a^2 - \dfrac{2r}{a}} \\ \text{and } x = p + a \pm \sqrt{p^2 + q - a^2 + \dfrac{2r}{a}} \end{matrix}\right\} \text{where}$$

a^2 is equal to the Root of the Cubic,

$y^3 -$

$$y^3 \left.\begin{matrix}-p^2\\-q\end{matrix}\right\} y^2 \left.\begin{matrix}+2pr\\+s\end{matrix}\right\} y - r^2 = 0.$$

The Demonstration is deduced from the last Article, as that of 546 is from the preceding.

553. An ingenious Correspondent, Mr. SAMUEL COLE, who teaches a Mathematical School near *Horndean* in *Hampshire*, has obliged me with the following plain Demonstration of *Cardan*'s *Rules*, from their first Principles, as also an Improvement of the same, which here follow.

"There are two Forms of Cubic Equations, *viz.*

1. $x^3 + px = \pm q$.
2. $x^3 - px = \pm q$.

554. CASE I.

	1	$x^3 + px = + q$. Query x?
Suppose	2	$x = m - n$. Because x is here affirmative.
$2 \circledcirc^{3}$	3	$x^3 = m^3 - 3m^2n + 3mn^2 - n^3$
$2 \times \overline{p}$	4	$px = pm - pn$
$3 + 4$	5	$\left\{\begin{matrix}x^3 + px = m^3 - 3m^2n + 3mn^2 - n^3\\ + pm - pn = q.\end{matrix}\right.$
Again, Suppose	6	$3mn = p$. (Or, $p - 3mn = 0$) [Also $n = \frac{\frac{1}{3}p}{m}$
Then $6 \times \overline{m-n}$	7	$3mmn - 3mnn = pm - pn$
7 equated to (0)	8	$0 = -3m^2n + 3mn^2 + pm - pn$
$5 - 8$	9	$(x^3 + px =)\ m^3 - n^3 = q$
But	10	$3mn = p$. By Supposition, Step 6th.
$10 \div 3m$	11	$n = \frac{p}{3m}$
$11 \circledcirc^{3}$	12	$n^3 = \frac{p^3}{27m^3}$. (And $-n^3 = -\frac{p^3}{27m^3}$)
9, 12	13	$m^3 - (n^3 =)\ \frac{p^3}{27m^3} = q$. (Or, $m^3 - \frac{p^3}{27m^3} = q$.) By Substitution.
$13 \times m^3$	14	$m^6 - \frac{p^3}{27} = qm^3$

14 by Reduction	15	$m^6 - qm^3 = \frac{p^3}{27}$. A quadratic Equation.
15 C □	16	$m^6 - qm^3 + \frac{q^2}{4} = \frac{p^3}{27} + \frac{q^2}{4}$
16 ɯ^2	17	$m^3 - \frac{q}{2} = \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$
17 $+ \frac{q}{2}$	18	$m^3 = \frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$ A simple Cubic.
18 ɯ^3	19	$m = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$
But (9th Step)	20	$q = m^3 - n^3$
Subst. Value of m^3 from Step 18.	21	$\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}} - n^3 = q$
21 by Reduction.	22	$\left\{ \begin{array}{l} -n^3 = q - \frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}} \text{ Or, (because} \\ q - \frac{q}{2} = \frac{q}{2}) \end{array} \right.$
22 contracted	23	$-n^3 = \frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$ A simp. Cubic.
23 ɯ^3	24	$-n = -\sqrt[3]{\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$
19, 24	25	$\left\{ \begin{array}{l} m - n = (x =) \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \\ - \sqrt[3]{\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \end{array} \right.$ The Theor.

555. If the Equation be $x^3 + px = -q$, find the Root of $x^3 + px = +q$, as above, and change its Sign.

556. *Cardan*'s Rule, above given, requires two Extractions of the Cube Root, and therefore the following Method, which requires but *one*, is much easier.

See the 19th Step above, $m = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$

And from the 6th Step it appears that

$n = \frac{\frac{1}{3}p}{m} = \frac{\frac{1}{3}p}{\sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}}$ Which is an Improvement of *Cardan*'s first Rule.

557. CASE

557. CASE II.

$x^3 - px = \pm q$. This is demonstrated in the same Manner as before, by putting $x = m + n$, *&c.* Here also $n = \frac{\frac{1}{3}p}{m} = \frac{\frac{1}{3}p}{\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}}$, and x is found by one Extraction.

CHAP. XXVIII.

Of the NEWTONIAN METHOD *of* approximating *to the Roots of* NUMERICAL EQUATIONS.

558. WHEN any Equation is proposed to be resolved, if the Root lies between two Numbers, that differ from one another by Unity, you may conclude that Root is *incommensurable*, by what has been heretofore taught. The Root, however, may be approximated to what Degree of Exactness you please, by the following Method, first proposed by Sir *Isaac Newton.*

559. In the first Place you make Trial with such Numbers as you find (by the foregoing Articles) are near the Value of the Root, and you will soon get two Numbers, one of which will give the Equation a *positive*, and the other a *negative Value*, and consequently the true Root, which makes it vanish, or $= 0$, will be between these Numbers, and when it is incommensurate you may, by easy Trials, approach to it as near as can be necessary by Fractions either *vulgar*, or *decimal.*

560. But before we proceed any farther, it may be worth observing, *that any common Number expressed by the nine Digits, is but a simple, quadratic, cubic,* &c. *Equation,* whose Root is $10 = x$, and therefore not expressed, but only the Coefficients of each Term. Thus, for Example,

$$\text{The Number}\left\{\begin{array}{l} 96 = 9x + 6, \text{ a } \textit{Simple} \text{ Equation.} \\ 596 = 5x^2 + 9x + 6, \text{ a } \textit{Quadratic.} \\ 7596 = 7x^3 + 5x^2 + 9x + 6, \text{ a } \textit{Cubic.} \\ 17596 = x^4 + 7x^3 + 5x^2 + 9x + 6, \text{ a } \textit{Biquadratic.} \end{array}\right.$$

Where, for the Sake of expeditious Operation, we abbreviate thoſe Equations into the Expreſſion of their Coefficients only, by calling x *Ten*, x^2 an *Hundred*, x^3 a *Thouſand*, *&c.* which as they conſiſt of Cyphers only, with Unit prefixed, *viz.* 10, 100, 1000, *&c.* can be eaſily underſtood without being expreſſed in the common Form of an *Equation* or *Series.* But to proceed.

561. Let the Equation propoſed be $x^2 - 6x\ 7 = 0$, if we ſuppoſe $x = 2$, the Reſult is $4 - 12 + 7 = -1$, which being negative, and the Suppoſition of $x = 0$ giving a poſitive Reſult, it follows that the Root is betwixt 0 and 2. Next, we ſuppoſe $x = 1$; whence, $x^2 - 6x + 7 = 1 - 6 + 7 = +2$, which being poſitive, we infer the Root is betwixt 1 and 2, and conſequently incommenſurable. In order to approximate to it, we ſuppoſe $x = 1\frac{1}{2}$, and find $x^2 - 6x + 7 = 2\frac{1}{4} - 9 + 7 = \frac{1}{4}$; and this Reſult being poſitive, we infer the Root muſt be betwixt 2 and $1\frac{1}{2}$. And therefore we try $1\frac{3}{4}$, and find $x^2 - 6x + 7 = \frac{49}{16} - \frac{42}{4} + 7 = 3\frac{1}{16} - 10\frac{8}{16} + 7 = -\frac{7}{16}$, which is negative; ſo that we conclude the Root to be betwixt $1\frac{3}{4}$ and $1\frac{1}{2}$. And therefore we try next $1\frac{5}{8}$, which giving alſo a negative Reſult, we conclude the Root is betwixt $1\frac{1}{2}$ (or $1\frac{4}{8}$) and $1\frac{5}{8}$. We try therefore $1\frac{9}{16}$, and the Reſult being poſitive, we conclude that the Root muſt be betwixt $1\frac{9}{16}$ and $1\frac{10}{16}$, and therefore is nearly $1\frac{19}{32}$.

562. Or you may approximate more eaſily by transforming the Equation propoſed into another whoſe Roots ſhall be equal to 10, 100, or 1000 Times the Root of the former (by Inſt. 503.) and taking the Limits greater in the ſame Proportion. This Transformation is eaſy; for you are only to multiply the 2d Term by 10, 100, or 1000, the third Term by their Squares, the 4th by their Cubes, *&c.* The Equation of the laſt Example is thus transformed into $x^2 - 600x + 70000 = 0$, whoſe Roots are 100 Times the Roots of the propoſed Equation, and whoſe Limits are 100 and 200. Proceeding as before, we try 150, and find $x^2 - 600x + 70000 = 22500 - 90000 + 70000 = 2500$, ſo that 150 is leſs than the Root. You next

try

try 175, which, giving a negative Result must be greater than the Root: and thus proceeding you find the Root to be betwixt 158 and 159: from which you infer, that the least Root of the proposed Equation $x^2 - 6x + 7 = 0$ is betwixt 1.58 and 1.59, being the hundredth Part of the Root of $x^2 - 600x + 70000 = 0$.

563. If the Cubic Equation $x^3 - 15x^2 + 63x - 50 = 0$ is proposed to be resolved, by substituting 0 for x the Value of $x^3 - 15x^2 + 63x - 50$ is negative, and by substituting 3 for x, that Quantity becomes positive. $x = 1$ gives it negative, and $x = 2$ gives it positive, so that the Root is between 1 and 2, and therefore incommensurable. You may proceed as in the foregoing Examples to approximate to the Root. But there are other Methods by which you may do that more easily and readily; which we proceed to explain.

564. When you have discovered the Value of the Root to less than an Unit (as in this Example you know it is a little above 1) suppose the Difference betwixt its real Value and the Number that you have found nearly equal to it, to be represented by f: As in this Example.

Let $x = 1 + f$. Substitute this Value for x in the Equation, thus,

$$\begin{aligned} x^3 &= \quad 1 + 3f + 3f^2 + f^3 \\ -15x^2 &= -15 - 30f - 15f^2 \\ +63x &= \quad 63 + 63f \\ -50 &= -50 \end{aligned}$$

$$x^3 - 15x^2 + 63x - 50 = -1 + 36f - 12f^2 + f^3 = 0.$$

Now because f is supposed less than Unit, its Powers f^2, f^3, may be neglected in this Approximation; so that assuming only the two first Terms, we have $-1 + 36f = 0$, or $f = \frac{1}{36} = .027$; so that x will be nearly 1.027.

565. You may have a nearer Value of x by considering, that seeing $-1 + 36f - 12f^2 + f^3 = 0$, it follows that

$$f = \frac{1}{36 - 12f + f^2} = \text{(by substituting } \tfrac{1}{36} \text{ for } f\text{) nearly}$$

$$= \frac{1}{36 - 12 \times \frac{1}{36} + \frac{1}{36} \times \frac{1}{36}} = \tfrac{1296}{46225} = .02803.$$

566. But the Value of f may be corrected and determined more accurately by supposing g to be the Difference betwixt its real Value, and that which we last found nearly equal to it. So that $f = .02803 + g$. Then by substituting this Value for f in the Equation

$f^3 - 12f^2 + 36f - 1 = 0$, it will stand as follows,

$$\left.\begin{array}{l} f^3 = 0.0000220226 + 0.002357g + 0.08409g^2 + g^3 \\ -12f^2 = -.00942816 \quad -0.67272g - 12g^2 \\ +36f = 1.00908 \quad + 36g \\ -1 = -1. \end{array}\right\} = 0$$

$$= -0.0003261374 + 35.329637g - 11.9195g^2 + g^3 = 0.$$

Of which the two first Terms, neglecting the rest, give $35.329637 \times g = 0.0003261374$, and $g = \frac{.0003261374}{35.329637} = 0.00000923127$. So that $f = 0.02803923127$; and $x = 1 + f = 1.02803923127$; which is very near the true Root of the Equation that was proposed.

If still a greater Degree of Exactness is required, suppose h equal to the Difference betwixt the true Value of g and that we have already found, and proceeding as above you may correct the Value of g.

567. For another Example; let the Equation to be resolved be $x^3 - 2x - 5 = 0$, and by some of the preceding Methods you discover one of the Roots to be between 2 and 3. Therefore you suppose $x = 2 + f$, and substituting this Value for it, you find,

$$\left.\begin{array}{l} x^3 = 8 + 12f + 6f^2 + f^2 \\ -2x = -4 - 2f \\ -5 = -5 \end{array}\right\} = 0 =$$

$$= -1 + 10f + 6f^2 + f^3;$$

from which we find that

10f

$10f = 1$ nearly, or $f = 0.1$. Then to correct this Value, we suppose $f = 0.1 + g$, and find

$$\left.\begin{array}{rcl} f^3 = & 0.001 + & 0.03g + 0.3g^2 + g^3 \\ 6f^2 = & 0.06 + & 1.2\ g + 6.\ g^2 \\ 10f = & 1. + & 10.\ g \\ -1 = & -1. & \end{array}\right\} = 0$$

$$= 0.061 + 11.23g + 6.3g^2 + g^3,$$

$$\text{so that } g = \frac{-0.061}{11.23} = -0.0054.$$

Then by supposing $g = -.0054 + h$, you may correct its Value, and you will find that the Root required is nearly 2.09455147.

568. "In all these Operations, you will approximate sooner to the Value of the Root, if you take the three last Terms of the Equation, and extract the Root of the Quadratic Equation consisting of these three Terms."

Then, in Inst. 564, instead of the two last Terms of the Equation $f^3 - 12f^2 + 36f - 1 = 0$, if you take the three last and extract the Root of the Quadratic $12f^2 - 36f + 1 = 0$, you will find $f = .028031$, which is much nearer the true Value than what you discover by supposing $36f - 1 = 0$.

It is obvious, that this Method extends to all Equations.

569. "By assuming Equations affected with general Coefficients, you may, by this Method, deduce general Rules or *Theorems* for approximating to the Roots of proposed Equations of whatever Degree."

Let $f^3 - pf^2 + qf - r = 0$ represent the Equation by which the Fraction f is to be determined, which is to be added to the Limit, or subtracted from it, in order to have the near Value of x.

Then $qf - r = 0$ will give $f = \frac{r}{q}$. But since $f = \frac{r}{f^2 - pf + q}$, by substituting $\frac{r}{q}$ for f, we have this Theorem for finding f nearly, *viz.*

$f =$

$$f = \frac{r}{\frac{r^2}{q^2} - \frac{pr}{q} + q} = \frac{q^2 \times r}{q^3 - pqr + r^2}.$$

570. After the same Manner, if it is a Biquadratic, by which f is to be determined, as $f^4 - pf^3 + qf^2 - rf + s = 0$, then f being very little, we shall have $f = \frac{s}{r}$; which Value is corrected by considering that $f = \frac{s}{r - qf + pf^2 - f^3} =$ (by substituting $\frac{s}{r}$ for f) $= \frac{s}{r - \frac{qs}{r} + \frac{ps^2}{r^2} - \frac{s^3}{r^3}}$, whence we have this Theorem for all Biquadratic Equations,

$$f = \frac{r^3 \times s}{-s^3 + ps^2r - qsr^2 + r^4}.$$

571. Other Theorems may be deduced by assuming the three Terms of the Equation, and extracting the Root of the Quadratic which they form.

Thus to find the Value of f in the Equation $f^3 - pf^2 + qf - r = 0$ where f is supposed to be very little, we neglect the first Term f^3, and extract the Root of the Quadratic $pf^2 - qf + r = 0$, or of $f^2 - \frac{q}{p} \times f + \frac{r}{p} = 0$; and we find $f = \frac{q}{2p} \pm \sqrt{-\frac{r}{p} + \frac{q^2}{4p^2}} = \frac{q \pm \sqrt{q^2 - 4pr}}{2p}$ nearly.

But this Value of f may be corrected by supposing it equal to m, and substituting m^3 for f^3 in the Equation $f^3 - pf^2 + qf - r = 0$, which will give $m^3 - pf^2 + qf - r = 0$, and $pf^2 - qf + r - m^3 = 0$; the Resolution of which Quadratic Equation gives $f = \frac{q \pm \sqrt{q^2 - 4pr + 4pm^3}}{2p}$, very near the true Value of f.

After the same Manner you may find like Theorems for the Roots of Biquadratic Equations, or of Equations of any Dimention whatever.

572. In

572. In general, let $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$, &c. $+ A = 0$ represent an Equation of any Dimensions n, where A is supposed to represent the absolute known Term of the Equation. Let k represent the Limit next less than any of the Roots, and supposing $x = k + f$, substitute the Powers of $k+f$ instead of the Powers of x, and there will arise, $\overline{k+f}^n + p \times \overline{k+f}^{n-1} + q \times \overline{k+f}^{n-2} + r \times \overline{k+f}^{n-3}$, &c. $+ A = 0$, or by Involution, disposing the Terms according to the Dimensions of f . . . .

$$\left.\begin{array}{l}
k^n + nk^{n-1} \times f + n \times \frac{n-1}{2} k^{n-2} f^2 +, \text{\&c.} \\
pk^{n-1} + p \times \overline{n-1}\, k^{n-2} \times f + p \times \overline{n-1} \times \frac{n-2}{2} k^{n-3} f^2 +, \text{\&c.} \\
qk^{n-2} + q \times \overline{n-2}\, k^{n-3} \times f + q \times \overline{n-2} \times \frac{n-3}{2} k^{n-4} f^2 +, \text{\&c.} \\
rk^{n-3} + r \times \overline{n-3}\, k^{n-4} \times f + r \times \overline{n-3} \times \frac{n-4}{2} k^{n-5} f^2 +, \text{\&c.} \\
\text{\&c.}
\end{array}\right\} + A = 0,$$

where

where neglecting all the Powers of f after the first two Terms, you find

$$f = \frac{-A - k^n - p\,k^{n-1} - q\,k^{n-2} - r\,k^{n-3}, \text{ \&c.}}{nk^{n-1} + p \times \overline{n-1}\,k^{n-2} + q \times \overline{n-2}\,k^{n-3} + r \times \overline{n-3}\,k^{n-4}, \text{ \&c.}}$$

Then $k + f = x$, the Root, which by this Series may be had to any required Exactness in any Equation of the Power n.

573. "By this Method you may discover Theorems for approximating to the Roots of *pure Powers*;" as to find the n Root of any Number A; suppose k to be the nearest less Root in Integers, and that $k + f$ is the true Root, then shall $k^n + n\,k^{n-1} f + n \times \frac{n-1}{2} k^{n-2} f^2$, &c. $= A$; and assuming only the two first Terms, $f = \frac{A - k^n}{n\,k^{n-1}}$; Or, more nearly, taking the three first Terms,

$$f = \frac{A - k^n}{n\,k^{n-1} + n \times \frac{n-1}{2} k^{n-2} f}, \text{ and (taking } \frac{A - k^n}{n\,k^{n-1}} = f\text{)}$$

$$f = \frac{A - k^n}{n\,k^{n-1} + \frac{nn-n}{2} k^{n-2} \times \frac{A-k^n}{nk^{n-1}}} = \frac{A - k^n}{nk^{n-1} + \frac{n-n}{2k} \times \overline{A - k^n}}$$

$= (\text{putting } m = A - k^n) = \frac{k\,m}{n\,k^n + \frac{n-1}{2} \times m}$; which is a *rational* Theorem for approximating to f.

574. You may find an *irrational* Theorem for it by assuming the three first Terms of the Power of $k + f$, *viz.* $k^n + n\,k^{n-1} f + n \times \frac{n-1}{2} k^{n-2} f^2 = A$.

For, $n\,k^{n-1} f + n \times \frac{n-1}{2} k^{n-} f^2 = A - k^n = m$; and resolving this Quadratic Equation, you find

$f =$

$$f = -\frac{k}{n-1} \pm \sqrt{\frac{2m}{n \times \overline{n-1} \times k^{n-2}} + \frac{k^2}{\overline{n-1}|^2}} =$$

$$= -\frac{k}{n-1} \pm \sqrt{\frac{2mn - 2m + nk^n}{n \times \overline{n-1}|^2 k^{n-2}}}.$$

575. In the Application of these Theorems, when a near Value of f is obtained, then adding it to k substitute the Aggregate in Place of k in the Formula, and you will, by a new Operation, obtain a more correct Value of the Root required; and, by thus proceeding, you may arrive at any Degree of Exactness.

Thus, to obtain the Cube Root of 2, suppose $k = 1$, and $f\left(= \frac{km}{nk^n + \frac{n-1}{2}m}\right) = \frac{1}{4} = 0.25$. In the second Place, suppose $k = 1.25$, and f will be found by a new Operation, equal to 0.009921, and consequently, $\sqrt[3]{2} = 1.259921$ nearly. By the irrational Theorem, the same Value is discovered for $\sqrt[3]{2}$.

CHAP. XXIX.

Of the Method of Series *by which you may approximate to the Roots of* Literal *Equations.*

576. IF there be only two Letters, x and a, in the proposed Equation, suppose a equal to Unit, and find the Root of the numeral Equation that arises from the Substitution, by the Rules of the last Chapter. Multiply these Roots by a, and the Products will give the Roots of the proposed Equation.

Thus the Roots of the Equation $x^2 - 16x + 55 = 0$ are found to be 5 and 11. And therefore the Roots of the Equation $x^2 - 16ax + 55a^2 = 0$, will be $5a$ and $11a$. The Roots of the Equation $x^3 + a^2x - 2a^3 = 0$ are found

by enquiring what are the Roots of the numeral Equation $x^3 + x - 2 = 0$, and since one of these is 1, it follows, that one of the Roots of the proposed Equation is a; the other two are *imaginary*.

577. If the Equation to be resolved involves more than two Letters, as $x^3 + a^2 x - 2a^3 + ayx - y^3 = 0$, then the Value of x may be exhibited in a Series having its Terms composed of the Powers of a and y with their respective Coefficients; which will *converge the sooner the less* y *is in respect of* a, *if the Terms are continually multiplied by the Powers of* y, *and divided by those of* a. Or, *will converge the sooner the greater* y *is in respect of* a, *if the Terms be continually multiplied by the Powers of* a, *and divided by those of* y. Since when y is very little in respect of a, the Terms y, $\frac{y^2}{a}$, $\frac{y^3}{a^2}$, $\frac{y^4}{a^3}$, $\frac{y^5}{a^4}$, &c. decrease very quickly. If y vanish in respect of a, the second Term will vanish in respect of the first, since $\frac{y^2}{a} : y :: y : a$. And after the same Manner $\frac{y^3}{a^2}$ vanishes in respect of the Term immediately preceding it.

578. But when y is vastly great in respect of a, than a is vastly great in respect of $\frac{a^2}{y}$, and $\frac{a^2}{y}$ in respect of $\frac{a^3}{y^2}$; so that the Terms a, $\frac{a^2}{y}$, $\frac{a^3}{y^2}$, $\frac{a^4}{y^3}$, $\frac{a^5}{y^4}$, &c. in this Case *decrease* very swiftly. In either Case, the Series converge swiftly that consist of such Terms; and a few of the first Terms will give a near Value of the Root required.

Example *of Case* I.

579. If a Series for x is required from the proposed Equation that shall converge the sooner the less y is in respect of a; to find the first Term of this Series, we shall suppose y to vanish; and extracting the Root of the Equation $x^3 + a^2 x - 2a^3 = 0$, consisting of the remaining Parts of the Equation that do not vanish with y, we find, (by Inst. 576.) thst $x = a$; which is the true Value of x when y vanishes, but is only near its Value when y does not vanish, but only is very little. To get a Value still nearly the true Value of x, suppose the Difference of a from the

the true Value to be p, or that $x = a + p$. And substituting $a + p$ in the given Equation for x, you will find,

$$\left.\begin{aligned} x^3 &= a^3 + 3a^2p + 3ap^2 + p^3 \\ +a^2x &= a^3 + a^2p \\ -2a^3 &= -2a^3 \\ +ayx &= a^2y + apy \\ -y^3 &= -y^3 \end{aligned}\right\} = 0; \text{ or}$$

$$x^3 + a^2x - 2a^3 + ayx - y^3 = \left.\begin{aligned} &4a^2p + 3ap^2 + p^3 \\ &a^2y + apy - y^3 \end{aligned}\right\} = 0.$$

But since, by Supposition, y and p are very little in respect of a, it follows that the Terms $4a^2p$, 2^2y, where y and p are separately of the *least* Dimensions, are vastly great in respect of the rest; so that, in determining a near Value of p, the rest may be neglected: And from $4a^2p + a^2y = 0$, we find $p = -\frac{1}{4}y$. So that $x = a + p = a - \frac{1}{4}y$, nearly.

Then to find a nearer Value of p, and consequently of x, suppose $p = -\frac{1}{4}y + q$, and substituting this Value for it in the last Equation, you will find,

$$\left.\begin{aligned} p^3 &= -\tfrac{1}{64}y^3 + \tfrac{3}{16}y^2q - \tfrac{3}{4}yq^2 + q^3 \\ 3ap^2 &= \tfrac{3}{16}ay^2 - \tfrac{3}{2}ayq + 3aq^2 \\ 4a^2p &= -a^2y + 4a^2q \\ ayp &= -\tfrac{1}{4}ay^2 + ayq \\ a^2y &= a^2y \\ -y^3 &= -y^3 \end{aligned}\right\} = 0 =$$

$$\left.\begin{aligned} = &-\tfrac{65}{64}y^3 + \tfrac{3}{16}y^2q - \tfrac{3}{4}yq^2 + q^3 \\ &-\tfrac{1}{16}ay^2 - \tfrac{1}{2}ayq + 3aq^2 \\ &+4a^2q \end{aligned}\right\} = 0.$$

And since, by the Supposition, q is very little in respect of p, which is nearly $= -\frac{1}{4}y$, therefore q will be very little in respect of y; and consequently all the Terms of the last Equation will be very little in respect of these two, *viz.* $-\frac{1}{16}ay^2$, $+4a^2q$, where y and q are of least Dimensions separately: Particularly the Term $-\frac{1}{2}ayq$ is little in respect of $4a^2q$, because y is very little in respect of a; and it is little in respect of $-\frac{1}{16}ay^2$ because q is little in respect of y.

Neglect therefore the other Terms, and supposing $-\frac{1}{16}ay^2$

 $+4a^2q$

$+ 4a^2 q = 0$, you will have $q = \frac{1}{64} \times \frac{y^2}{a}$; so that $x = a - \frac{1}{4}y + \frac{1}{64} \times \frac{y^2}{a}$. And by proceeding in the same Manner you will find $x = a - \frac{y}{4} + \frac{y^2}{64a} - \frac{131y^3}{512a^2} + \frac{509y^4}{16384a^3} -$, &c.

EXAMPLE *of Case* II.

580. When it is required to find a Series for x that shall converge sooner, the *greater* y is in respect of any Quantity a, you need only suppose a to be very little in respect of y, and proceed by the same Reasoning as in the last Example on the Supposition of y being very little.

Thus, to find a Value for x in the Equation $x^3 - a^2x + ayx - y^3 = 0$ that shall converge the sooner the greater y is in respect of a. Suppose a to vanish, and the remaining Terms will give $x^3 - y^3 = 0$, or $x = y$. So that when y is vastly great, it appears that $x = y$ nearly.

But to have the Value of x more accurately, put $x = y + p$, then,

$$\left.\begin{array}{r} x^3 = y^3 + 3y^2p + 3yp^2 + p^3 \\ -a^2x = -a^2y - a^2p \\ +ayx = ay^2 + ayp \\ -y^3 = -y^3 \end{array}\right\} = 0 =$$

$$= + 3y^2p + 3yp^2 + p^3 - a^2y - a^2p + ay^2 + ayp:$$

Where the Terms $3y^2p + ay^2$ become vastly greater than the rest, y being vastly greater than a or p; and consequently $p = -\frac{1}{3}a$ nearly.

Again, by supposing $p = -\frac{1}{3}a + q$ you will transform the last Equation into

$$\left.\begin{array}{l} -\frac{3}{27}a^3 + 3y^2q + 3yq^2 + q^3 \\ -a^2y - ayq - aq^2 \\ -\frac{2}{3}a^2q \end{array}\right\} = 0.$$

Where the two Terms $3qy^2 - a^2y$ must be vastly greater than any of the rest, a being vastly less than y, and q vastly less than

a, by the Suppoſition; ſo that $3qy^2 - a^2y = 0$, and $q = \frac{a^2}{3y}$ nearly. By proceeding in this Manner, you may correct the Value of y and find that

$$x = y - \tfrac{1}{3}a + \frac{a^2}{3y} + \frac{a^3}{81y^2} - \frac{8a^4}{243y^3}, \text{ \&c.}$$

Which Series converges the ſooner the greater y is ſuppoſed to be taken in reſpect of a.

581. In the Solution of the firſt Example thoſe Terms were always compared in order to determine p, q, r, &c. in which y and thoſe Quantities p, q, r, &c. were *ſeparately* of feweſt Dimenſions. But in the ſecond Example, thoſe Terms were compared in which a and the Quantities p, q, r, &c. were of leaſt Dimenſions ſeparately. And theſe always are the proper Terms to be compared together, becauſe they become vaſtly greater than the reſt, in the reſpective Hypotheſes.

582. *In general*; to determine the firſt, or any, Term in the Series, *ſuch Terms of the Equation are to be aſſumed together only*, as *will be found to become vaſtly greater than the other Terms*; that is, which give a Value of x which ſubſtituted for it in all the Terms of the Equation ſhall raiſe the Dimenſions of the other Terms all above, or all below, the Dimenſions of the aſſumed Terms, according as y is ſuppoſed to be vaſtly little, or vaſtly great in reſpect of a.

Thus to determine the firſt Term of a converging Series expreſſing the Value of x in the laſt Equation $x^3 - a^2x + ayx - y^3 = 0$, the Terms ayx and $-y^3$ are not to be compared together, for they would give $x = \frac{y^2}{a}$, which ſubſtituted for x the Equation becomes,

$\frac{y^6}{a^3} - ay^2 + y^3 - y^3 = 0$, where the firſt Term is of more Dimenſions than the aſſumed Terms ayx, $-y^3$: and the 2d of fewer; ſo that the two firſt Terms cannot be neglected in reſpect of the two laſt, neither when y is very great nor very little, compared with a. Nor are the Terms x^3, ayx, fit to be compared together in order to obtain the firſt Term of a Series for x, for the like Reaſon.

But

But x^3 may be compared with $-a^2x$, as alſo $-a^2x$ with $-y^3$, for that End. Theſe two give the firſt Terms of a Series that converges the ſooner the leſs y is; as $x^3 = y^3$ gives the firſt Term of a Series that converges the ſooner the greater y is. The laſt Series was given in the preceding Article. The comparing x^3 with $-a^2x$ gives theſe two Series,

$$x = a - \tfrac{1}{2}y - \frac{y^2}{8a} + \frac{7y^3}{16a^2} - \frac{59y^4}{128a^3}, \text{ \&c.}$$

$$x = -a + \tfrac{1}{2}y + \frac{y^2}{8a} + \frac{9y^3}{16a^2} + \frac{69y^4}{128a^3}, \text{ \&c.}$$

The comparing $-a^2x$ with $-y^3$ gives,

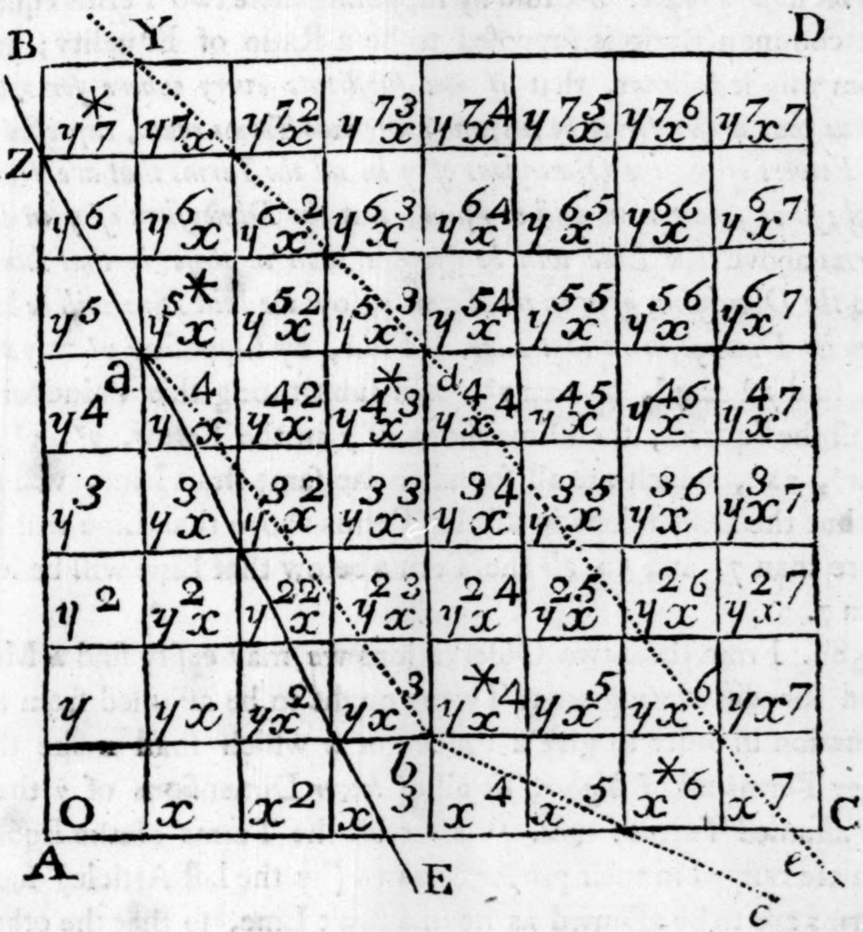
$$x = -\frac{y^3}{a^2} - \frac{y^4}{a^3} - \frac{y^5}{a^4} - \frac{y^6}{a^5} -, \text{ \&c.}$$

And theſe Series give three Values of x when y is very little; the laſt of which is itſelf alſo very little in that Caſe, as it appears indeed from the Equation, that when y vaniſhes, the three Values of x become $+a$, $-a$, and 0, becauſe when y vaniſhes, the Equation becomes $x^3 - a^2x = 0$, whoſe Roots are, a, $-a$, 0.

583. It appears ſufficiently from what we have ſaid, that when an Equation is propoſed involving x and y, and the Value of x is required in a converging Series, the Difficulty of finding the firſt Term of the Series is reduced to this; *to find what Terms aſſumed in order to determine a Value of* x *expreſſed in ſome Dimenſions of* y *and a will give ſuch a Value of it as ſubſtituted for it in the other Terms will make them all of more Dimenſions of* y, *or all of leſs Dimenſions of* y, *than thoſe aſſumed Terms.*

To determine this, draw B A and A C at right Angles to each other, compleat the Parallelogram A B C D and divide it into equal Squares, as in the Figure. In theſe Squares place the Powers of x from A towards C, and the Powers of y from A towards B, and in any other Square place that Power of x that is directly below it in the Line A C, and that Power of y that is in a Parallel with it in the Line A B; ſo that the Index of x in any Square may expreſs its Diſtance from the Line A B, and

and the Index of y in any Square may express its Distance from the Line A C. Of this Square we are to observe,



584. 1. That the Terms are not only in Geometrical Progression in the Vertical Column A B, or the Horizontal A C, and their Parallels; but also in the Terms taken in any oblique strait Line whatever; for in any such Terms it is manifest that the *Indices* of y and x will be in arithmetical Progression. The Indices of y, because those Terms will remove equally from the Line A C, or approach equally to it, and the Indices of y in any such Terms are as their Distances from that Line A C. The Indices of x will also be in arithmetical Progression, because these Terms equally remove from, or approach to the Line A B. Thus for Example, in the Terms y^7, $y^5 x$, $y^3 x^2$, yx^3, the Indices of y decreasing by the common Difference 2, while the Indices of x increase in the Progression of the natural Numbers, the common Ratio of the Terms is $\frac{x}{y^2}$. It follows,

585. 2. From

585. 2. From the last Observation, that *if any two Terms be supposed equal, then all the Terms in the same strait Line with these Terms will be equal:* because by supposing these two Terms equal, the common Ratio is supposed to be a Ratio of Equality; and from this it follows, that *if you substitute every where for* x *the Value that arises for it by supposing any two Terms equal, expressed in the Powers of* y, *the Dimensions of* y *in all the Terms that are found in the same strait Line will be equal*; but *the Dimensions of* y *in the Terms* above *that Line will be* greater *than in those in that Line*; and *the Dimensions of* y *in the Terms* below *the said Line will be* less *than its Dimensions in that Line.* Thus, by supposing $y^7 = yx^3$, we find $x^3 = y^6$, or $x = y^2$; and substituting this Value for x in all the Squares, the Dimensions of y in the Terms, y^7, $y^5 x$, $y^3 x^2$, yx^3, which are all found in the same strait Line, will be 7, but the Dimensions in all the Terms above that Line will be more than 7, and in all the Terms below that Line will be less than 7.

586. From these two Observations we may easily find a Method for discovering what Terms ought to be assumed from an Equation in order to give a Value for x which shall make the other Terms all of *higher*, or all of *lower* Dimensions of y than the assumed Terms: *viz.* "after all the Terms of the Equation are ranged in their proper Squares (by the last Article) such Terms are to be assumed as lie in a strait Line, so that the other Terms either lie all above the strait Line, or fall all below it."

For Example, suppose the Equation proposed is $y^7 - ay^5 x + y^4 x^3 + a^2 y x^4 - ax^6 = 0$, then marking with an Asterisk the Squares in the last Article which contain the same Dimensions of x and y as the Terms in the Equation, imagine a Ruler ZE to revolve about the first Square at Z marked at y^7, and as it moves from A towards C, it will first meet the Term $ay^5 x$, and while the Ruler joins these two Terms, all the other Terms lie above it: from which you infer that by supposing these Terms equal, (*viz.* $y^7 = ay^5 x$) you shall obtain a Value of x, which substituted for it, will give all the other Terms of higher Dimensions of y, than those Terms: And hence we conclude that the Value of x deduced from supposing these Terms equal, *viz.* $\frac{y^2}{a}$, is the first Term of a Series that will converge the sooner the less y is in respect of a.

587, If

587. If the Ruler be made to revolve about the same Square, at Y, the contrary Way from D towards C, it will first meet the Term $y^4 x^3$, and by supposing $y^7 + y^4 x^3 = 0$, we find $y = x$, which gives the first Term of a Series for x, that converges the sooner the greater that y is. And this is the celebrated Rule invented by Sir *Isaac Newton*, for this Purpose.

588. This Rule may be extended to Equations having Terms that involve Powers of x and y with *Fractional* or *Surd* Indices; "by taking Distances from A in the Lines AC and AB proportional to these Fractions and Surds," and thence determining the Situation of the Terms of the proposed Equation in the Parallelogram ABCD.

589. It is to be observed also, that when the Line joining any two Terms has all the other Terms on one Side of it, by them you may find the first Term of a converging Series for x, and thus "various such Series can be deduced from the same Equation." As, in the last Example, the Line ab joining $y^5 x$ and yx^4 has all the Terms above it; and therefore supposing $-ay^5 x + a^2 y x^4 = 0$ we find $x^3 = \frac{y^4}{a}$, and $x = \frac{y^{\frac{4}{3}}}{a^{\frac{1}{3}}}$, which is the first Term of another converging Series for x. Again, the strait Line bc joining yx^4 and x^6 has all the other Terms above it, and therefore, supposing $a^2 y x^4 - ax^6 = 0$, we find $ay = x^2$, and $x = a^{\frac{1}{2}} y^{\frac{1}{2}}$, the first Term of another Series for x, converging also the sooner the less y is. There are two Series converging the sooner the greater y is, to be deduced from supposing $y^7 = -y^4 x^3$, by the Line Yd, or $y^4 x^3 = ax^6$, by the Line de. And, to find all these Series, "describe a *Polygon* Zabed, having a Term of the Equation in each of its Angles, and including all the other Terms within it, then a Series may be found for x, by supposing any two Terms equal that are placed in any two adjacent Angles of the Polygon."

590. If the Ruler ZE be made to move parallel to itself, all the Terms which it will touch at once will be of the same Dimensions of y: for they will bear the same Proportion to one another as the Terms in the Line ZE themselves. The Terms which the Ruler will touch first will have fewer Dimensions of y, than those it touches afterwards in the Progress of its Motion, if it moves towards D; but more Dimensions than they, if it

moves towards A. The Terms in the ſtrait Line ZE, ſerve to determine the firſt Term of the converging Series required.

Theſe with the Terms it touches afterwards ſerve to determine the ſucceeding Terms of the converging Series; all the reſt vaniſhing compared with theſe, when y is very little and the Ruler moves from A towards D, or when y is vaſtly great and the Ruler moves from D towards A.

591. The ſame Author* gives another Method for diſcovering the firſt Term of a Series that ſhall converge the ſooner the leſs y is. "Suppoſe the Term where y is ſeparately of feweſt Dimenſions to be Dy^l; compare it ſucceſſively with the other Terms, as with $Ey^m x^s$, and obſerve where $\frac{l-m}{s}$ is found *greateſt*; and putting $\frac{l-m}{s} = n$, Ay^n will be the firſt Term of a Series that ſhall converge the ſooner the leſs y is:" For in that Caſe Dy^l and $Ey^m x^s$ will be infinitely greater than any other Terms of the propoſed Equation. Suppoſe $Fy^e x^k$ is any other Term of the Equation, and, by the Suppoſition, $\frac{l-m}{s}$ $(= n)$ is greater than $\frac{l-e}{k}$, and conſequently, multiplying by k, you find nk greater than $l-e$, and $nk+e$ greater than l; now if for x you ſubſtitute Ay^n, then $Fy^e x^k = FA^k y^{nk+e}$, which therefore will vaniſh compared with Dy^l (ſince $nk+e$ is greater than l) when y is infinitely little. Thus therefore all the Terms will vaniſh compared with Dyl and $Ey^m x^s$ which are ſuppoſed equal; and conſequently they will give the firſt Term of a Series that will converge the ſooner the leſs y is.

592. If you obſerve "when $\frac{l-m}{s}$ is found *leaſt* of all, and ſuppoſe it equal to n, then will Ay^n be the firſt Term of a Series that will converge the ſooner the *greater* y is." For in that Caſe Dy^l and $Ey^m x^s$ will be infinitely greater than $Fy^e x^k$, becauſe $\frac{l-m}{s}$ $(= n)$ being leſs than $\frac{l-e}{k}$, it follows that nk is leſs than $l-e$, and $nk+e$ leſs than l, and conſequently $Fy^e x^k$ $(= FA^k y^{nk+e})$ vaſtly leſs than Dy^l, when y is very great.

593. After

* Sir *I. Newton.*

593. After the same Manner, if you compare any Term $Dy^l x^b$, where both x and y are found, with all the other Terms, and observe where $\frac{l-m}{s-b}$ is found the *greatest* or *least*, and suppose $\frac{l-m}{s-b} = n$, then may Ay^n be the first Term of a converging Series. For supposing that $Fy^e x^k$ is any other Term of the Equation, if $\frac{l-m}{s-b} (= n)$ is greater than $\frac{l-e}{k-b}$, then shall $nk - nb$ be greater than $l - e$, and $nk + e$ greater than $l + nb$. But $nk + e$ are the Dimensions of y in $Fy^e x^k$ when $x = Ay^n$, and $l + nb$ are the Dimensions of y in $Ey^m x^s$; therefore $Fy^e x^k$ is of more Dimensions of y than $Ey^m x^s$, and therefore vanishes compared to it when y is supposed infinitely little. In the same Manner, if $\frac{l-m}{s-b}$ is less than $\frac{l-e}{k-b}$, then will $Ey^m x^s$ be infinitely greater than $Fy^e x^k$, when y is infinite.

594. When the first Term (Ay^n) of the Series is found by the preceding Method, then by supposing $x = Ay^n + p$, and substituting this Binomial and its Powers for x and its Powers, there will arise an Equation for determining p the second Term of the Series. This new Equation may be treated in the same Manner as the Equation of x, and by the Rule of Inst. 586, the Terms that are to be compared in order to obtain a near Value of p, may be discovered; by Means of which Terms p may be found: Which suppose equal to By^{n+r}, then by supposing $p = By^{n+r} + q$, the Equation may be transformed into one for determining q the third Term of the Series, and by proceeding in the same Manner you may determine as many Terms of the Series as you please; finding $x = Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. where the Dimensions of y ascend or descend according as r is positive or negative; and always "in Arithmetical Progression, that this Value of x being substituted for it in the proposed Equation, the Terms involving y and its Powers may fall in with one another, so that more than one may always involve the same Dimension of y, which may mutually destroy each other and make the whole Equation vanish, as it ought to do."

595. It is obvious that as the Dimensions of y in $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, *&c.* are in an Arithmetical Progression whose Difference is r, the Square, Cube, or any Power s of $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ *&c.* will consist of Terms wherein the Dimensions of y will constitute an Arithmetical Progression having the same common Difference r; for these Dimensions will be sn, $sn + r$, $sn + 2r$, $sn + 3r$, &c. Therefore, if in any Term $Ey^m x^s$ you substitute for x the Series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$ *&c.* the Terms of the Series expressing $Ey^m x^s$ will consist of these Dimensions of y, viz. $m + sn$, $m + sn + r$, $m + sn + 2r$, $m + sn + 3r$ *&c.* and by a like Substitution in any other Term as $Fy^e x^k$, the Dimensions of y will be $e + nk$, $e + nk + r$, $e + nk + 2r$, $e + nk + 3r$ *&c.* The former Series of Indices must coincide with the latter Series, that the Terms in which they are found may be compared together, and be found equal with opposite Signs so as to destroy one another, and make the whole Equation vanish.

596. The first Series consists of Terms arising by adding some Multiple of r to $m + sn$, the latter by adding some Multiple of r to $e + nk$; and that these may coincide, some Multiple of r added to $m + sn$ must be equal to some other Multiple of r added to $e + nk$. From which it appears, that the Difference of $m + sn$ and $e + nk$ is always a Multiple of r; and consequently that r is a Divisor of the Difference of Dimensions of y in the Terms $Ey^m x^s$ and $Fy^e x^k$, supposing $x = Ay^n$. It follows therefore "that r is a common Divisor of the Differences of the Dimensions of y in the Terms of the Equation, when you have substituted Ay^n for x in all the Terms." And if r be assumed equal to the *greatest* common Divisor (excepting some Cases afterward to be mentioned) you will have the true Form of a Series for x. And now the Dimensions y^n, y^{n+r}, y^{n+2r}, y^{n+3r} *&c.* being known, there remains only, by Calculation to determine the general Coefficients A, B, C, D, &c. in order to find the Series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. $= x$.

597. This leads us to Sir *Isaac Newton*'s *Second General Method* of Series; which consists in assuming a Series with undetermined Coefficients expressing x, as $Ay^n + By^{n+r} + Cy^{n+2r} +$

&c.

&c. Where *A*, *B*, *C*, &c. are suppoſed as yet unknown, but n and r are diſcovered by what we have already demonſtrated; and ſubſtituting this every where for x, you muſt ſuppoſe, in the new Equation that ariſes, the Sum of all the Terms that involve the ſame Dimenſion of y to vaniſh, by which Means you will obtain particular Equations, the *firſt* of which will give *A*, the *ſecond* *B*, the *third* *C*, &c. and theſe Values being ſubſtituted in the aſſumed Series for *A*, *B*, *C*, &c. the Series for x will be obtained as far as you pleaſe.

598. Let us apply, for Example, this Method to the Equation (of Inſt. 577.) $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$. Suppoſe it is required to find a Series converging the ſooner the leſs y is: Its firſt Term (by Inſt. 579, or 583) is found to be a, ſo that $n = 0$. Subſtitute a for x in the Equation, and the Terms become $a^3 + a^3 - 2a^3 + a^2y - y^3$, and the Differences of the Indices are 0, 1, 2, 3; whoſe greateſt common Meaſure is 1, ſo that $r = 1$. Aſſume therefore $x = A + By + Cy^2 + Dy^3$ &c. and ſubſtitute this Series for x in the Equation. Then

$$\begin{aligned}
x^3 &= A^3 + 3A^2By + 3AB^2y^2 + B^3y^3 + \&c.\\
&\qquad\qquad + 3A^2Cy^2 + 3A^2Dy^3 + \&c.\\
&\qquad\qquad\qquad + 6ABCy^3 + \&c.\\
+a^2x &= a^2A + a^2By + a^2Cy^2 + a^2Dy^3 + \&c.\\
+ayx &= \qquad aAy + aBy^2 + aDy^3 + \&c.\\
-2a^3 &= -2a^3\\
-y^3 &= \ldots\ldots\ldots -1 \times y^3.
\end{aligned}$$

599. Now ſince $x^3 + a^2x + ayx - 2a^3 - y^3 = 0$, it follows that the Sum of theſe Series involving y muſt vaniſh. But that cannot be if the Coefficient of every particular Term does not vaniſh. For every Term where y is infinitely little, is infinitely greater than the following Terms, ſo that if every Term does not vaniſh of itſelf, the Addition or Subtraction of the following Terms which are infinitely leſs than it, or of the preceding Terms which are infinitely greater, cannot deſtroy it; and therefore the whole cannot vaniſh. It appears therefore that $A^3 + a^2A - 2a^3 = 0$, is an Equation for determining *A*, and gives $A = a$.

In order to determine *B*, you muſt ſuppoſe the Sum of the Coefficients affecting y to vaniſh, *viz.* $\overline{3A^2B + a^2B + aA} \times y = 0$, or ſince $A = a$, $4a^2By + a^2y = 0$, and $B = -\frac{1}{4}$.

To

To determine C, in the same Manner suppose $3AB^2y^2 + 3A^2Cy^2 + a^2Cy^2 + aBy^2 = 0$, or substituting for A and B their Values already found, $\frac{3ay^2}{16} + 4a^2Cy^2 - \frac{ay^2}{4} = 0$, and consequently $C = \frac{1}{64a}$. And, by proceeding in the same Manner, $D = \frac{131}{512a^2}$, so that $x = a - \frac{1}{4}y + \frac{1}{64a}y^2 + \frac{131}{512a^2}y^3$, &c. as we found before in Inst. 579.

600. By this Method you may transfer Series from one undetermined Quantity to another, and obtain Theorems for the *Reversion* of Series.

Suppose that $x = ay + by^2 + cy^3 + dy^4 +$ &c. and it is required to express y by a Series consisting of the Powers of x. It is obvious that when x is very little, y is also very little, and that in order to determine the first Term of the Series, you need only assume $x = ay$. And therefore $y = \frac{x}{a}$; so that $n = 1$. By substituting $\frac{x}{a}$ for y, you find the Dimensions of x in the Terms will be 1, 2, 3, 4, &c. so that $r = 1$ also. You may therefore assume $y = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. And by the Substitution of this Value of y you will find,

$$\begin{array}{llll} ay = & aAx + & aBx^2 + & aCx^3 + \text{\&c.} \\ by^2 = & & bA^2x^2 + & 2bABx^3 + \text{\&c.} \\ cy^3 = & & & cA^3x^3 + \text{\&c.} \\ \text{\&c.} & & & \text{\&c.} \end{array}$$

But the first Term being already found to be $\frac{x}{a}$, you have $A = \frac{1}{a}$; and since $aB + bA^2 = 0$, it follows that $B = -\frac{b}{a^3}$. After the same Manner you will find $C = \frac{2b^2 - ac}{a^4}$. Whence $y = \frac{x}{a} - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^4}x^3 +$ &c.

601. Suppose

601. Suppose again you have $ax + bx^2 + cx^3 + dx^4 +$ &c. $= gy + hy^2 + iy^3 + ky^4$ &c. to find x in Terms of y. You will easily see, (by Inst. 586.) that the first Term of the Series for x is $\frac{gy}{a}$, that $n = 1$, $r = 1$. Therefore assume $x =$ $Ay + By^2 + Cy^3$ &c. and by substituting this Value for x and bringing all the Terms to one Side, you will have

$$\begin{array}{llll}
ax = aAy + aBy^2 & + aCy^3 & + \text{\&c.} \\
bx^2 = & bA^2y^2 & + 2bABy^3 & + \text{\&c.} \\
cx^3 = & & cA^3y^3 & + \text{\&c.} \\
\text{\&c.} & & \text{\&c.} & \\
-gy = -gy & & & \\
-hy^2 = & \ldots -hy^2 & & \\
-iy^3 = & & \ldots\ldots -iy^3 & \\
\text{\&c.} & & \text{\&c.} &
\end{array}$$

From whence we see, first, that $aA = g$, and $A = \frac{g}{a}$. 2°. That $aB + bA^2 - h = 0$, and $B = \frac{h}{a} - \frac{bg^2}{a^3}$. 3°. That $aC + 2bAB + cA^3 - i = 0$, and therefore $C = \frac{i - 2bAB - cA^3}{a}$. And thus the three first Terms of the Series $Ay + By^2 + Cy^3$ &c. are known.*

This Method is very general; but in some particular Cases a Difficulty will arise, which Mr. *Maclaurin* has shewn how to avoid, to whose Treatise of Algebra we must refer the Reader, as it would be very tedious here to dwell on that Subject.

* See Mr. *De Moivre* in *Phil. Trans.* 240.

CHAP.

CHAP. XXX.

Mr. De Moivre's *Theorem for raising an Infinite Series to any given Power. Also the Evolution or Reversion of Series deduced from thence.*

602. IN the Philosophical Transactions,* the late Mr. *De Moivre* has given us the following excellent Theorem for raising any Multinomial or infinite Series to any given Power, *viz.*

$$\overline{az+bz^2+cz^3+dz^4+ez^5+fz^6, \&c.}^{\,m} \text{ will be}$$

$$= a^m z^m$$

$$+\frac{m}{1} a^{m-1} b z^{m+1}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2} a^{m-2} b^2 \\ +\frac{m}{1} a^{m-1} c \end{array}\right\} z^{m+2}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3} a^{m-3} b^3 \\ +\frac{m}{1}\times\frac{m-1}{1} a^{m-2} bc \\ +\frac{m}{1} a^{m-1} d \end{array}\right\} z^{m+3}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4} a^{m-4} b^4 \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1} a^{m-3} b^2 c \\ +\frac{m}{1}\times\frac{m-1}{1} a^{m-2} bd \\ +\frac{m}{1}\times\frac{m-1}{2} a^{m-2} c^2 \\ +\frac{m}{1} a^{m-1} e \end{array}\right\} z^{m+4}$$

$+$

* N°. 240.

$$\left.\begin{array}{r}
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}a^{m-5}b^5 \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}a^{m-4}b^3c \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^2d \\
+\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{2}a^{m-3}bc^2 \\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-2}be \\
\frac{m}{1}+\frac{m-1}{1}a^{m-2}cd \\
\times\frac{m}{1}a^{m-1}f
\end{array}\right\} z^{m+5}$$

$$\left.\begin{array}{r}
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}a^{m-6}b^6 \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{1}a^{m-5}b^4c \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}a^{m-4}b^3d \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}\times\frac{m-3}{2}a^{m-4}b^2c^2 \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^2e \\
+\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{1}a^{m-3}bcd \\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-3}bf \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}c^3 \\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-2}ce \\
+\frac{m}{1}\times\frac{m-1}{2}a^{m-2}d^2 \\
\frac{m}{1}a^{m-1}g
\end{array}\right\} z^{m+6}, \text{ \&c.}$$

603. For

603. For underſtanding of which, it is only neceſſary to conſider all the Terms by which the ſame Power of z is multiplied: In order to which two things in each of theſe Terms muſt be conſidered, 1°, The Product of certain Powers of the given Quantities of Coefficients a, b, c, d, &c. And, 2°, The *Unciæ* or Products of $\frac{m}{1} \times \frac{m-1}{2}$, *&c.* prefixed to them.

Now to find all the Products belonging to the ſame Power of z. For Example; to find that Product whoſe Index is $m+r$ (r being any whole Number) the ſaid Products muſt be diſtinguiſhed into ſeveral Claſſes. Thoſe which immediately after ſome certain Power of a (by which all theſe Products begin) are Products of the firſt Claſs: As $a^{m-4}b^5e$ is a Product of the firſt Claſs, becauſe b immediately follows a^{m-4}. Thoſe which immediately after ſome Power of a have c, are Products of the ſecond Claſs. So $a^{m-1}ccd$ is a Product of the ſecond Claſs: Thoſe which immediately after ſome Power of a have d, are Products of the third Claſs, and ſo of the reſt.

This being underſtood, 1°, Multiply all the Products belonging to z^{m+r-1} (which immediately precedes z^{m+r}) by b, and divide them all by a. 2°, Multiply by c, and divide by a, all the Products belonging to z^{m+r-2} except thoſe of the firſt Claſs. 3°, Multiply by d, and divide by a all the Products belonging to z^{m+r-3} except thoſe of the firſt and ſecond Claſs. 4°, Multiply by e, and divide by a, all the Terms belonging to z^{m+r-4}, except thoſe of the firſt, ſecond, and third Claſs; and ſo on, till you meet twice with the ſame Term. Laſtly, add the Product of a^{m-r} into the Letter whoſe Exponent is $r-1$ to all theſe Terms.

Note, The Exponent of a Letter is the Number expreſſing what Place that Letter has in the Alphabet, as is 3 the Exponent of the Letter c, being the 3d Letter.

By this Rule it is manifeſt that it is eaſy to find all the Products belonging to the ſeveral Powers of z, if you have but the Product belonging to zm, viz. a^m.

604. Next to find the *Unciæ* prefixed to every Product, you muſt conſider the Sum of the Units contained in the Exponents of the Letters that compoſe it (the Index of a excepted); then I write as many Terms of the Series $m \times m-1 \times m-2 \times$

$m-3$,

$m - 3$, *&c.* as there are Units in the Sum of these Indexes; this Series is to be the Numerator of a Fraction, whose Denominator is the Product of the several Series $1 \times 2 \times 3 \times 4 \times 5$, *&c.* $1 \times 2 \times 3 \times 4 \times 5$, *&c.* $1 \times 2 \times 3 \times 4 \times 5 \times 6$, *&c.* the first of which contains as many Terms as there are Units in the Index of b; the second as many as there are Units in the Index of c; the third as many as there are Units in the Index of d, &c.

The Demonstration of this see in the above cited Transaction.

Here follows an Example or two of the Use of this Theorem.

EXAMPLE I.

605. To raise this infinite Series $\frac{1}{x} + \frac{1}{xx} + \frac{1}{x^3} + \frac{1}{x^4}$, *&c.* to the second Power, or to square it.

In this Case in the Theorem $m = 2$, $z = x$, $a = \frac{1}{xx}$, $b = \frac{1}{x^4}$, $c = \frac{1}{x^6}$, $d = \frac{1}{x^8}$, *&c.* therefore $\overline{\frac{1}{x} + \frac{1}{xx} + \frac{1}{x^3} + \frac{1}{x^4}}\,^2$, *&c.* will be $\frac{1}{xx} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{4}{x^5}$, *&c.* for the first Term $a^m z^m \left(= \frac{1}{x^4} \times xx\right)$ is $= \frac{1}{xx}$. The second Term $\frac{m}{1} a^{m-1} b z^{m+1} \left(= \frac{2}{xx} \times \frac{1}{x^4} \times x^3\right)$ is $= \frac{2}{x^3}$.

The Third $\left.\begin{matrix} \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 \\ + \frac{m}{1} a^{m-1} c \end{matrix}\right\} z^{m+2} \left\{\begin{matrix} = 2 \times \frac{1}{2} \times \frac{1}{x^8} \times x^4 \\ = \frac{1}{x^4}, \end{matrix}\right.$

and $2 \times \frac{1}{xx} \times \frac{1}{x^6} \times x^4$ is $\left\{ = \frac{2}{x^4} \right.$ $\Bigg\} = \frac{3}{x^4}$.

The Fourth Term $\left.\begin{matrix} \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 \\ \frac{m}{1} + \frac{m-1}{1} a^{m-2} b c \\ \frac{m}{1} a^{m-1} d \end{matrix}\right\} z^{m+3} = \left\{ \vphantom{\begin{matrix}a\\a\\a\end{matrix}} \right.$

 2 ×

$$\left.\begin{matrix} 2 \times \frac{1}{2} \times \frac{0}{3} \times xxx \times \frac{1}{x^{12}} \times x^5 = 0 \\ 2 \times 1 \times 1 \times \frac{1}{x^4} \times \frac{1}{x^6} \times x^5 = \frac{2}{x^5} \\ 2 \times \frac{1}{xx} \times \frac{1}{x^8} \times x^5 = \frac{2}{x^5} \end{matrix}\right\} \text{is} = \frac{4}{x^5}.$$

EXAMPLE II.

606. To square this infinite Series $1 - x + x^2 + x^2 + x^4$, &c.

In this Case in the Theorem $m = 2$, $z = x$. $a = \frac{1}{x} - 1$. $b = 1$. $c = -1$. $d = 1$, &c. and so $\overline{1 - x + x^2 - x^3 + x^4, \&c.}^{\,2}$ will be $= 1 - 2x + 3x^2 - 4x^3 + 5x^4$, &c. for $a^m z^m$ $\left(= \overline{\frac{1}{x} - 1}^{\,2} \times x^2\right) = 1 - 2x + xx$. The Second Term $\frac{m}{1} a^{m-1} b z^{m+1}$ $\left(= \frac{2}{1} \times \overline{\frac{1}{x} -}^{\,1} \times x^3\right) = 2xx - 2x^3$.

The Third Term $\left.\begin{matrix} \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 \\ \frac{m}{1} a^{m-1} c \end{matrix}\right\} z^{m+2} = \left\{\begin{matrix} \frac{2}{1} \times \frac{1}{2} \times \overline{\frac{1}{x} - 1}^{\,2-2} \times 1 \times x^4 \\ \frac{2}{1} \times \overline{\frac{1}{x} - 1}^{\,1} \times -1 \times x^4 \end{matrix}\right\} = -x^3 + 2x^4$, &c.

EXAMPLE III.

607. To raise $1 - x + x^3 - x^5 + x^7$, &c. to the third Power, or to cube it.

Here $m = 3$. $z = x$. $a = \frac{1}{x} - 1$, $b = 0$, $c = 1$, $d = 0$. and so the third Power will be $1 - 3x + 3x^2 + 2x^3 - 6x^4$, &c. for $a^m z^m$ $\left(= \overline{\frac{1}{x} - 1}^{\,3} \times x^3\right) = 1 - 3x + 3xx - x^3$.

$$\frac{m}{1}a^{m-1}bz^{m+1}\left(=3\times\overline{\frac{1}{x}-1}^{\,3-1}\times 0\times x^4\right)=0.$$

$$\left.\begin{array}{r}\frac{m}{1}\times\frac{m-1}{2}a^{m-2}b^2\\ \frac{m}{1}a^{m-1}c\end{array}\right\}z^{m+2}=\left\{\begin{array}{l}\\ \\ \end{array}\right.$$

$$\left.\begin{array}{l}3\times 1\times\overline{\frac{1}{x}-1}^{\,1}\times 0^2\times x^5=9\\ 3\times\overline{\frac{1}{x}-1}^{\,2}\times 1\times x^5=3x^3-6x^4\times 3x^5\end{array}\right\}=$$

$3x^3-6x^4+3x^5$.

And because $b=0$, and also d, therefore the next Term of the general Theorem will be 0. And thus you may proceed on.

Example IV.

608. To extract the Root of an infinite Series; that is, if z be $=ax+bx^2+cx^3+dx^4+ex^5$, &c. to find the Value of x in an infinite Series of Terms affected with z, and free from x.

First, Let us suppose $x=fz+hz^2+kz^3+lz^4+mz^5+nz^6$, &c. Then by the Theorem $x^2=f^2z^2+2fhz^3$

$$\begin{array}{l}+h^2z^4+2hkz^5+k^2z^6\\ +2fkz^4+2flz^5+2hlz^6\\ \qquad\qquad\qquad\qquad+2fmz^6\\ \qquad\qquad\qquad\qquad\qquad\&c.\end{array}$$

$$x^3=f^3z^3+3f^2hz^4+3fh^2z^5+h^3z^6,\ \&c.$$
$$+3f^2kz^5+3f^2lz^6$$
$$+6fhkz^6$$
$$x^4=f^4z^4+4f^3hz^5+6f^2h^2z^6,\ \&c.$$
$$x^5=f^5z^5+5f^4hz^6$$
$$x^6=f^6z^6,\ \&c.$$

Now substitute these Values in the Equation $0=-z+ax+bx^2+cx^3+dx^4+ex^5$, &c. and then will $-z=-z$.

$$+ax=+afz+ahz^2+akz^3+alz^4+amz^5+anz^6,\ \&c.$$
$$+bx^2=*bf^2z^2+2bfhz^3+bh^2z^4+2bflz^5+bk^2z^6,$$
$$+2bhkz^4+$$
$$+2bhkz^5+2bfmz^6$$
$$+cx^3$$

$$+cx^3 = \quad * \quad * \quad +cf^3z^3 + 3cf^2hz^4 + cfh^2z^5 + 3cf^2lz^6$$
$$+3cf^2kz^5 + \quad \&c.$$
$$3cf^2lz^6$$
$$+6fhkz^6$$
$$dx^4 = \quad * \quad * \quad * \quad +dfz^4 + 4df^3z^5 + 6df^2h^2z^6$$
$$+4df^3kz^6$$
$$ex^5 = \quad * \quad * \quad * \quad * \quad +ef^5z^5 + 5ef^4hz^6$$

609. Now if the Sum of the Coefficients of every Term in this Equation be made equal to nothing, we may get the Values of the Coefficients f, h, k, l, m, n, thus. The Sum of the Coefficients of the first Term $\left.\begin{matrix}-1\\ af\end{matrix}\right\} z$ will be $af - 1$. Whence if $af - 1 = 0$, then will f be $= \frac{1}{a}$. In like Manner the Sum of the Coefficients of the second Term $\left.\begin{matrix}ah\\ bf^2\end{matrix}\right\} z^2$ will be $ah + bf^2$. Whence if $ah + bf^2 = 0$, there will arise $h = \frac{-bf^2}{a} = \frac{-b}{a^3}$. In like Manner the Sum of the Coefficients of the third Term made equal to 0 will be $ak + 2bfh + cf^3 = 0$. Whence $k = \frac{-2bfh - cf^3}{a} = \frac{2b^2 - ac}{a^5}$. Again, $al + bh + 2bfk + 3cf^2h + df^4 = 0$. Whence $l = \frac{-bh - 2bfk + 3cfh - df^4}{a} = \frac{-b^3}{a^7 - 4b^5} + \frac{a^7 + 2ba}{a^6 + 3bc} + \frac{a^6 - d}{a^5} = \frac{5abc - 5b^3 - a^2d}{a^7}$.

So likewise m is $= \frac{14b^4 + 6abd - 21ab^2c + 3a^2c^2 - a^3e}{a^9}$.

And $n = \frac{-42b^5 + 84ab^3c - 28a^7bc - 28a^2bd + 7a^3ed + 7a^3cd + 7a^3bea^4f}{a^{11}}$.

Whence at length substituting these Values of the Coefficients f, h, k, l, m, n, in the assumed Equation $x = fz + hz^2 + kz^3 + lz^4 + mz^5 + nz^6$, &c. and the Root sought will be

$x =$

$$x = \frac{z}{a} - \frac{bz^2}{a^3} + \frac{2b^2 - ac}{a^5}z^3 + \frac{5abc - 5b^3 - a^2d}{a^7}z^4 + \frac{14b^4 + 6a^2bd - 21ab^2c + 3a^2c^2 - a^3e}{a^9}z^5, \&c.$$

610. If there are any Terms wanting in the proposed Equation, it is plain that they will likewise be wanting in the Root. For Example: If z be $= ax + cx^3 + ex^5$, &c. then will $x = \frac{z}{a} - \frac{ac}{a^5}z^3 + \frac{3a^2c^2 - a^3e}{a^9}z^5$, &c. But let this suffice for the present; the Application of Algebra to the Solution of Problems in Geometry must be deferred till we have premised the Principles of that Science, which next follow in a Method not only new, but 'tis presumed much more concise and perspicuous than any Thing before published on this Subject.

INSTI-

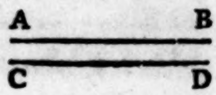
INSTITUTIONS OF GEOMETRY.

CHAP. I.

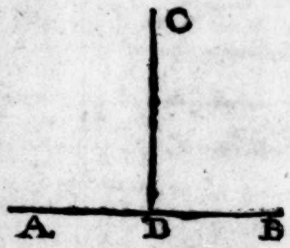
DEFINITIONS.

611. A POINT is that which hath no Parts; and may be repreſented to the Senſes by the Dot A.

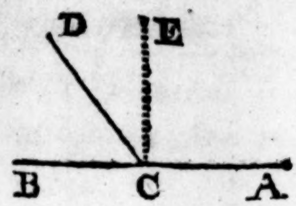
612. A LINE is deſcribed by the Motion of a Point; and is either a *Right Line* when the Motion is *ſtrait forwards* as A B: Or a *curved Line*, when the Motion is conſtantly directed to different Parts, as the Curve ABC. Lines are ſaid to be *Parallel*, when they are every where at an equal Diſtance, as AB, CD.

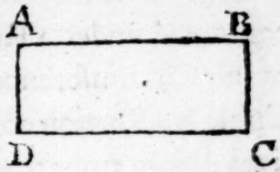


613. An ANGLE ABC is the Space contained between two Lines AB, CB meeting in the Point B. When a Right Line CD, ſtanding upon a Right Line AB, makes the Angles on either Side thereof equal to each other, *viz.* ADC = BDC, then both thoſe equal Angles are *Right Angles*; and the Right Line CD is called a *Perpendicular* to the Line AB.

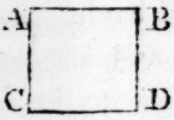


614. An OBTUSE ANGLE is greater than a *Right Angle*, as ACD is greater than ACE. And the Angle BCD, which is leſs than the *Right Angle* BCE is called an ACUTE ANGLE.

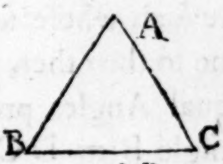


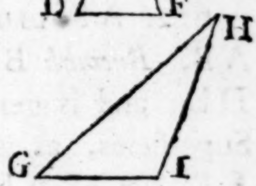


615. A SUPERFICIES is that which has *Length* and *Breadth*, as ABCD, and is generated by the Motion of a Line, as of AD or AB carried parallel to itself. All such whose 4 opposite Sides are equal, and *parallel*, are called *Parallelograms*. If all the Angles are Right, such a Figure is called a *Rectangle*.

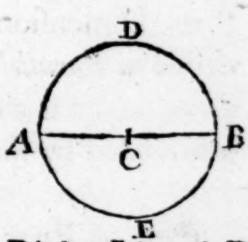


616. If all the 4 Sides are equal, and the Angle right, it is called a SQUARE, as ABCD. But if the Angles are not right, it is called a RHOMBUS, as EFGH. A Parallelogram whose Angles are not right, is called RHOMBOIDES, as IKLM. All other *four-sided Figures* are called TRAPEZIUMS, as the Figure NOPQ.



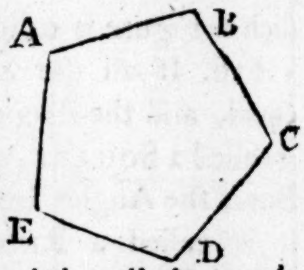


617. A Figure bounded by three Sides is called a TRIANGLE; if the three Sides be equal, it is said to be *Equilateral*, as ABC; if only *two are equal*, it has the Name *Isosceles*, as DEF; if the three Sides are all unequal, it is said to be *Scalenous*, as GHI. A Triangle is said to be *Right-angled*, that has one *Right Angle*; *Obtuse-angled*, if it has one Angle Obtuse; *Acute-angled*, when all the Angles are Acute; and *Equiangular*, when all the Angles are *equal*.

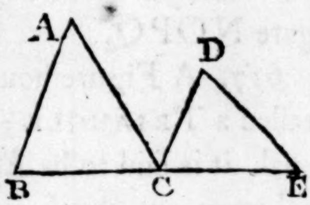


618. A CIRCLE is a Figure contained under one uniform Curve-line ADBE, which is called the CIRCUMFERENCE or PERIPHERY, described about the Point C, which is called the CENTER, and from which all Lines drawn to the Circumference are equal. The Right Line AB drawn through the Center, is called a *Diameter*; and divides the Circle into two equal Parts called SEMICIRCLES, as ADB and AEB.

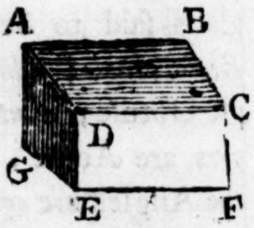
619. The SEGMENT of a Circle is a Figure contained under a Right-line E D, and a Portion of the Circumference E F D. A SECTOR of a Circle is a Portion contained between two Right-lines drawn from the Circumference to the Center, as A C B.



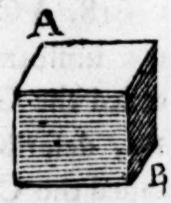
620. A *Pentagon* is a Figure contained under *five Sides*, and having as many Angles; if all the Sides and Angles are equal, the Pentagon is said to be *Regular*, as A B C D E, otherwise the Figure is said to be *Irregular*. In like Manner, if a Figure consist of Six equal Sides and Angles, it is called a *regular* HEXAGON; if of *Seven*, it is an HEPTAGON; if of Eight an OCTAGON, and so on.



621. SIMILAR Right-lined Figures are such whose several Angles are equal one to the other, and the Sides about the equal Angles proportional. Thus the Angle B = D C E, and the Sides A B : B C : : D C : C E. And the same is to be understood of the other Angles and Sides respectively.

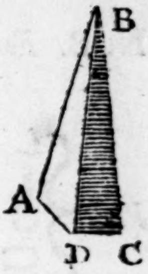


622. A SOLID is that which hath *Length*, A B, *Breadth* B C, and *Thickness* or *Depth* D E; and is generated by the Motion of a Superficies, as A B C D carried parallel to E F, or of A D E G to C F. When all these Dimensions are unequal the Solid is called a *Parallelopipedon*; but when they are all equal, it is called a CUBE, as A B, which is bounded by Six equal Squares.



623. A PRISM is a Solid contained under Planes, whereof the two End ones are equal, alike, and parallel; but the Sides are Parallelograms.

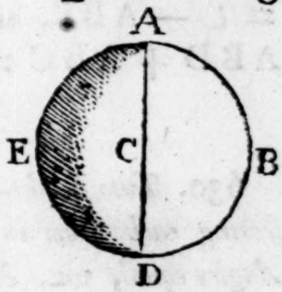
624. A PYRAMID is contained under divers triangular Planes set upon one Plane (called the *Base*) and terminating at the other End in one Point; as the solid ABCD.



625. A CONE is a Solid generated by the Revolution of a right-angled Triangle about one of the Sides, containing the Right-angle, which is called the *Axis* of the Cone; the other Side describes the circular *Base* of the Cone; while the slaunt Side subtending the Right-angle, describes the *Conic Superficies*. Thus ACD is the revolving Right-angled Triangle; and ADE the generated Cone.

626. A CYLINDER is a Solid generated by the Revolution of a right-angled Parallelogram ABCD about its Side AD, which makes the *Axis* of the Cylinder; while the too Sides AB and DC describe the circular Bases or Ends; and the Side BC the curved Surface thereof.

627. A SPHERE is a solid ABDE generated by the Revolution of Semicircle ABD about its Diameter AD (remaining fixed) which makes the *Axis* of the Sphere while the Circumference of the Semicircle describes the Convex Superficies of the Sphere.



628. *Similar Solids* are those which are contained under *similar Planes* equal in Number. Hence all *Cubes* and *Spheres* are *similar Solids*. And all *Cones*, and *Cylinders* are *similar*, whose *Axis* and *Diameters* of their Bases are *proportional.*

N. B. *The following Theorems do much depend upon the Axioms in* Inst. 198, 199, 200, 201, 202; *and therefore they ought to be well remembered by the Reader.*

N. B.

N. B. *For the Sake of Conciseness and Perspicuity, the following Symbols or Characters are used.*

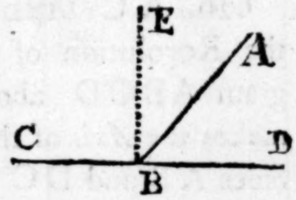
Viz.
- ∟ Denotes a *right Angle.*
- ∥ Parallel to.
- □ A Square.
- ▭ Parallelogram.
- △ A Triangle.

CHAP. II.

GEOMETRICAL THEOREMS.

THEOREM I.

629. *A Right-line* AB *standing upon another* CD, *makes Angles* ABC + ABD = 2 *Right-angles.*

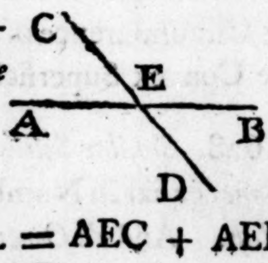


DEMONSTRATION.

If ABC = ABD, then is each a Right-angle by (613). If they are unequal; let BE be perpendicular to CD; then is ABD = ∟ − ABE, and ABC = ∟ + ABE, therefore (175) ABD + ABC = 2 ∟. *Q. E. D.*

THEOREM II.

630. *Two Right-lines* AB, CD, *intersecting each other in* E, *make the two opposite Angles equal, viz.* AED = CEB.

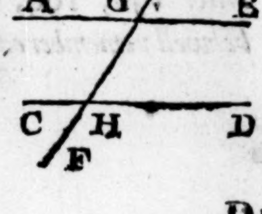


DEMONSTRATION.

The Angle AEC + CEB = (629) 2 ∟ = AEC + AED; therefore BEC = AED (200). *Q. E. D.*

THEOREM III.

631. *A Right-line* EF *cutting two Parallel Lines* AB, CD, *makes the alternate Angles equal, viz.* AGH = [illegible]GHD.



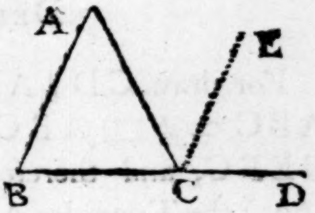
DE-

Demonstration.

Thus $AGE + AGH = (629)\ 2\,L = CHG + GHD$; but because of parallel Lines, it is $AGE = CHG$; therefore $AGH = GHD$. *Q. E. D.*

Theorem IV.

632. *The outward Angle* ACD *of any Triangle* ABC, *is equal to the two internal opposite Angles* A *and* B.



Demonstration.

Let CE be parallel to AB; then is the Angle $A = ACE$ by (631); and the Angle $B = ECD$, because of $AB \parallel EC$; therefore $A + B = ACE + ECD = ACD$. *Q. E. D.*

Theorem V.

633. *The three Angles of every plain Triangle* ABC, *are equal to two Right-angles.*

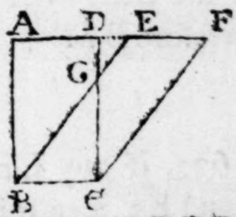
Demonstration.

For $ACD + ACB = 2\,L$, by (629); and $ACD = A + B$, by (632); therefore $ACB + A + B = 2\,L$. *Q. E. D.*

Theorem VI.

634. *Parallelograms* ABCD, BEFC, *which stand upon the same Base* BC, *and between the same Parallels* AF, BC, *are equal.*

Demonstration.

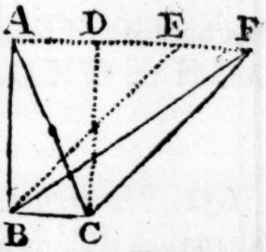


For by Definition $AD = BC = EF$; add the common Part DE to both, and we have $AE = DF$ (199); but $AB = DC$, and $BE = CF$; therefore is the Triangle $ABE = DCF$; take away the common Triangle DGE, there will remain the Trapezium $ABGD = CGEF$; to each of these add the Triangle BCG, and it makes the Parallelogram $ABCD = BEFC$. *Q. E. D.*

The-

Theorem VII.

635. *Triangles* ABC, BFC, *standing upon the same Base* BC, *and between the same Parallels* AF *and* BC, *are equal to one another.*



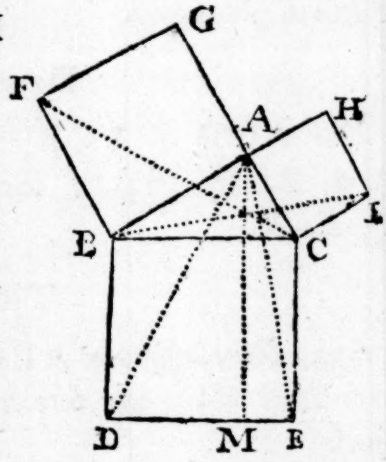
Demonstration.

For draw CD ∥ AB, and BE ∥ CF; then is the Triangle ABC $= \frac{1}{2} \square$ ABCD, and the Triangle BCF $= \frac{1}{2} \square$ BEFC, and therefore since the Parallelograms are equal by (634) the Triangles ABC, BCF will be so too. *Q. E. D.*

Theorem VIII.

636. *In a Right-angled Triangle* ABC, *the Square* BE *made upon the Side* BC *subtending the Right-angle* BAC, *is equal to the Sum of the Squares* BG, CH, *which are made of the Sides* AB *and* AC.

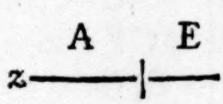
Demonstration.



Join AE, and AD, and draw AM ∥ CE, because the Angle DBC = FBA (616), add to both the common Angle ABC, then is the Angle ABD = FBC. Again, AB = FB, and BD = BC, (616); therefore the Triangle ADB = FBC. But the Parallelogram BM = 2 ABD; and the Parallelogram BG = 2 FBC, as is evident from the Figure to (635). Therefore is the $\square$ BM = BG the Square of the Side AB. In the same Manner, it is shewn that the $\square$ MC = CH; therefore BM + CM = BE = BG + CH. *Q. E. D.*

Theorem IX.

637. If any Line z be divided into two Parts A and E; then it is $zz = A^2 + E^2 + 2AE$.



DEMONSTRATION.

This is evident from (186). For $z = A + E$; and $zz = \overline{A+E} \times \overline{A+E} = A^2 + 2AE + E^2$. *Q. E. D.*

THEOREM X.

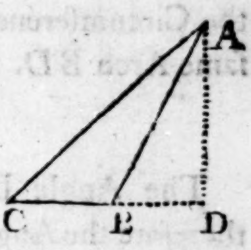
638. I say moreover, it is $z^2 + E^2 = A^2 + 2zE$.

DEMONSTRATION.

For to the above Equation $z^2 = A^2 + 2AE + E^2$, add on both Sides E^2, and it is $z^2 + E^2 = A^2 + 2AE + 2E^2 = A^2 + 2E \times \overline{A+E} = A^2 + 2EZ$. *Q. E. D.*

THEOREM XI.

639. In an obtuse angled Triangle ABC, having let fall the Perpendicular AD to the Base CB, continued out, it is $AC^2 = CB^2 + AB^2 + 2CB \times BD$.

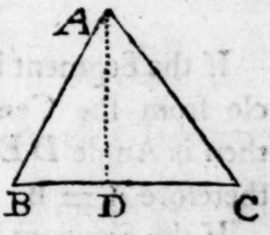


DEMONSTRATION.

For $AC^2 =$ (636.) $CD^2 + AD^2 =$ (637.) $CB^2 + 2\overline{BC \times BD} + BD^2 + AD^2 =$ (636.) $CB^2 + 2CB \times BD + AB^2$. *Q. E. D.*

THEOREM XII.

640. In an acute angled Triangle ABC, having let fall the Perpendicular AD on the Base BC, it is $AC^2 + BC^2 = AB^2 + 2BC \times CD$.



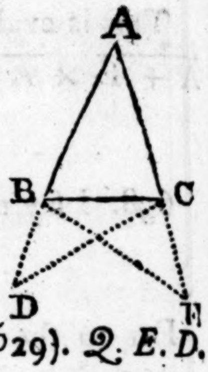
DEMONSTRATION.

For $AC^2 + BC^2 =$ (636.) $AD^2 + DC^2 + BC^2 =$ (638.) $AD^2 + BD^2 + \overline{2BC \times CD} =$ (636.) $AB^2 + \overline{2BC \times CD}$. *Q. E. D.*

THEOREM XIII.

641. In an Isosceles Triangle ABC, the equal Sides AB, AC, subtend equal Angles ABC, ACB at the Base BC.

Demonstration.



Produce AC, AB, and make AE = AD; and join BE, CD; then because AC = AB, and AE = AD and the common Angle A, the Triangles ABE, ACD will be equal, and the Angle E = D; and the Base BE = CD; also EC = BD, by Construction, therein the Triangles BEC, BDC, the Angle ECB = DBC; and consequently the Angle ACB = ABC by (629). *Q. E. D.*

Theorem XIV.

642. In a Circle ABDE, the Angle BCD, at the Center C, is double of the Angle A at the Circumference, when they stand upon the same Arch BD.

Demonstration.

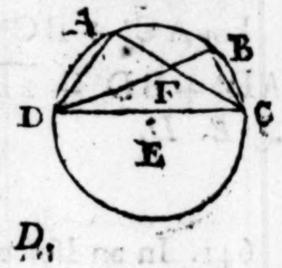
The Angle DCB = A + B; but AC = CB (by 618); therefore the Angle A = B (by 641); consequently, the Angle DCB = 2 A. *Q. E. D.*

Theorem XV.

643. Any two Angles DAC, DBC, in the same Segment of a Circle DABC, are equal.

Demonstration.

If the Segment be greater than a Semicircle from the Center E, draw ED, EC; then is Angle DEC = (642.) 2 A = 2 B, therefore A = B. *Q. E. D.*



If the Segment be less than a Semicircle; then in the Triangles DAF, BDF, the Sum of the Angles in each is the same (633). From each of which, take away the Angle AFD = BFC (650.) and ADB = BCA, by the first Case of this. There will remain the Angle DAC = DBC. *Q. E. D.*

The-

Theorem XVI.

644. The two opposite Angles D and B of any four-sided Figure ABDC described in a Circle; are equal to two Right Angles.

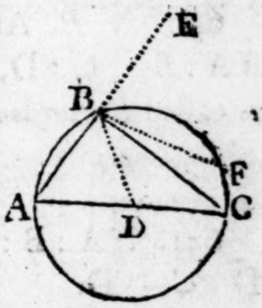
Demonstration.

Draw AC and BD; the Angle ABC + BCA + BAC = 2 L (633). But BDA = BCA (643), and also BDC = BAC; therefore ABC + ADC = 2 L. *Q. E. D.*

Theorem XVII.

645. The Angle ABC in a Semicircle is a Right one.

Demonstration.



From the Center D draw DB; because DB = DA, the Angle A = DBA, and the Angle DCB = DBC (641); therefore the Angle ABC = B + ACB = EBC (632); wherefore the Angles ABC and EBC are *Right-ones*, by (613). *Q. E. D.*

646. *Corol.* Hence the Angle A in the greater Segment BAC, is less than a Right Angle, because A + C = L (by 633 and 645); and the Angle ABF in the lesser Segment ABF, is greater than a Right Angle ABC.

Theorem XVIII.

647. If four Quantities A, B, C, D, are *directly* proportional, *viz.* A : B :: C : D, they will be so *alternately*, viz. A : C :: B : D, as also *inversely*, viz. B : A :: D : C. For in all these Cases, the Products of *Extremes and Means are equal*, viz. AD = BC, which could not be if the Terms in each Case were not proportional by (322).

Theorem XIX.

648. If A : B :: C : D, then by *Composition* of Ratios, it is A + B : B :: C + D : D.

Demonstration.

For since A : B :: C : D, it is AD = BC (322); add on both Sides the Quantity DB, and the Equation is AD + DB = BC + DB, which gives this Analogy, A + B : B :: C + D : D. *Q. E. D.*

Theorem XX.

649. If A : B :: C : D, then by Division of Ratios, it is A — B : B :: C — D : D.

Demonstration.

For since A : B :: C : D, it is AD = BC (322); from each Side subduct DB, and the Residue is AD — DB = BC — DB; wherefore A — B : B :: C — D : D. *Q. E. D.*

650. *N. B.* After the same Manner it may be shewn, that if A : B :: C : D, it is also A : B ± A :: C : C ± D, which is called *Conversion of Ratios.*

Theorem XXI.

651. If A : B :: C : D, it will be A + B : A — B :: C + D : C — D.

Demonstration.

For AD = BC (322) and 2 AD = 2 BC; to which on each Side add AC — DB, and we have AC + 2 AD — DB = AC + 2 BC — DB, that is, AC + CB — AD — DB = AC + AD — CB — DB, that is, $\overline{A + B} \times \overline{C - D} = \overline{A - B} \times \overline{C + D}$. Consequently A + B : A — B :: C + D : C — D. *Q. E. D.*

Theorem XXII.

652. If there be two Rows of Magnitudes $\left\{\begin{array}{l}\text{A, B, C, \&c.}\\ \text{R, S, T, \&c.}\end{array}\right.$

which taken ordinately two and two, are in the same Ratio, *viz.* $\left\{\begin{array}{l}\text{A : B :: R : S}\\ \text{B : C :: S : T, \&c.}\end{array}\right.$

They shall be also in the same Ratio by Equality, *viz.* $\{\text{A : C :: R : T.}$

De-

DEMONSTRATION.

For we have $AS = BR$, and $BT = CS$ by Hypothesis. Then $AS \times BT = BR \times CS$; divide each Side by BS, there remains $AT = RC$, which gives $A : C :: R : T$. *Q. E. D.*

Corol. Hence also $A : R :: C : T$.

THEOREM XXIII.

653. Ratios $A : B$ and $C : D$, which are the same to any third Ratio $x : y$, are the same to one another.

DEMONSTRATION.

Since $A : B :: x : y$, it is $Ay = Bx$ (322); also because $C : D :: x : y$, it is $Cy = Dx$; and therefore $Ay \times Dx = Cy \times Bx$; divide each Part by yx and there remains $AD = CB$; whence $A : B :: C : D$. *Q. E. D.*

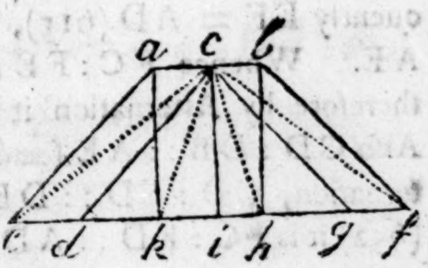
THEOREM XXIV.

654. The Ratio of $A \times B$ is to that of $A \times C$ the same with the Ratio of B to C, or $AB : AC :: B : C$. For the Product of the Extremes and Means are equal, *viz.* $AB \times C = AC \times B$. See (322).

THEOREM XXV.

655. Parallelograms, *acde*, *bcgf*, which have the same Height *ic*, have the same Ratio to one another as their Bases, *ed*, *gf*, have.

DEMONSTRATION.



The ▭ *acde* = ▭ *acik*, and *cbgf* = *cbhi*, by (634). Let $bh = ci = ak = A$; $ih = gf = B$; and $ki = ed = C$. Then is the ▭ $cbhi = A \times B$, and the ▭ $acik = A \times C$. Therefore (654) $cbhi : acik :: AB :: AC : B : C :: ih : ik :: gf : ed : cbfg : acde$. *Q. E. D.*

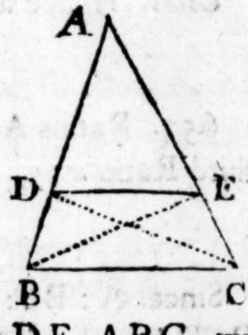
Corol. The same thing is true of the Triangles *cfg* and *ced*, as being the Halves of the Parallelograms, and which is evident also by (635).

THE-

Theorem XXVI.

656. If in the Triangle ABC, you draw DE ∥ BC, then will the Sides AB, AC, be cut proportionally, or it will be AD : DB :: AE : EC.

Demonstration.



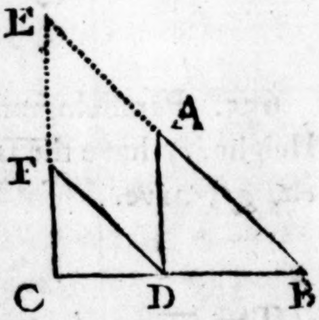
Join BE, and DC; because the Triangle DEB = DEC by (635), we have the Triangle ADE : DBE :: ADE : ECD; but (by 655) the Triangle ADE : DBE :: AD : DB; and ADE : DEC :: AE : EC. Therefore (653) it is AD : DB :: AE : EC. *Q. E. D.*

Corol. Hence because of similar Triangles ADE, ABC, we have AD : AE :: AB : AC; and therefore AB : AC :: DB : EC.

Theorem XXVII.

657. In equiangular Triangles ABD, FDC, the Sides about the equal Angles are proportional.

Demonstration.



Let their Bases CD and DB make a right Line CB; and produce AB and FC till they meet in E; because the Angle C = ADB, therefore CE ∥ AD by (631). Also because the Angle CDF = ABD, it is AB ∥ FD; therefore AEFD is a Parallelogram, and consequently EF = AD (615), and FD = AE. Whence FC : FE (= AD) :: CD : DB by (656); therefore by Alternation it is FC : CD :: AD : DB (647). Also CD : DB :: AE (= FD) AB; whence again by Alternation, CD : FD :: DB : AB. Wherefore by Equality (652) it is FC : FD :: AD : AB. *Q. E. D.*

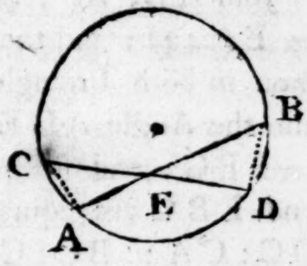
Theorem XXVIII.

658. If in a Circle two right Lines AB, CD, intersect each other in E, the Rectangle AE × EB shall be equal to the Rectangle CE × ED.

De-

DEMONSTRATION.

Join A C and D B; then the Angle C E A = B E D (630); alſo the Angle C = B (643); therefore the Triangles AEC, and DEB are equiangular (633); whence it is CE : EA :: EB : ED (657); conſequently CE × ED = AE × EB (322). *Q. E. D.*

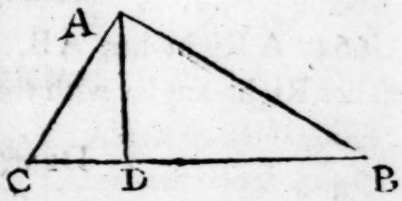


THEOREM XXIX.

659. If from the Right-angle A, of a right-angled Triangle B A C, be drawn the Perpendicular A D to the Baſe, then will the Triangles A D B and A D C be ſimilar both to the whole Triangle, and to each other.

DEMONSTRATION.

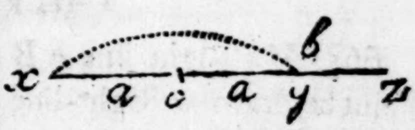
Becauſe the Angle BAC = ADB = L (613); and B common to both, the Triangles A D B and BAC are ſimilar, by (621, 657); for the ſame Reaſon the Triangles A D C and B A C are ſimilar. Conſequently the Triangles A D C, A D B, are ſimilar to each other. *Q. E. D.*



660. *Corol.* Hence it is B D : D A :: D A : D C by (621). Alſo B C : A C :: A C : D C; and C B : B A :: B A : B D.

THEOREM XXX.

661. If any Right-line xy be biſected in c, ſo that $xc = cy = a$, and another Right-line $yz = b$ be added thereto; then $\overline{2a + b} \times b + aa = aa + 2ba + bb = \overline{a + b}^2$.



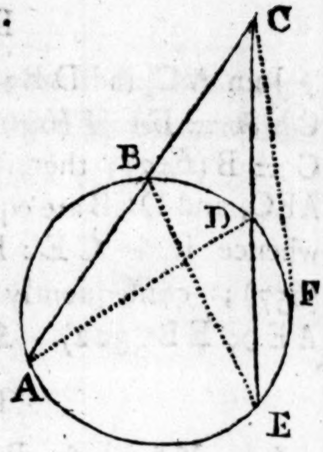
DEMONSTRATION.

This is evident from (186), or thus $\overline{xz + yz} \times yz + cy^2 = \overline{cy + yz}^2$.

THEOREM XXXI.

662. If from any Point C without a Circle, are drawn two Right-lines C A, C E, cutting the Circle in B and D, then it will be A C × B C = E C × D C.

DE-



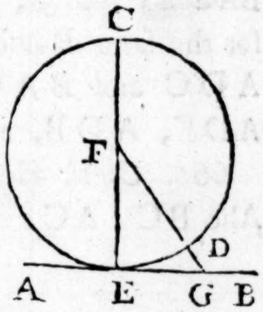
Demonstration.

Join AD, BE; then is the Angle A $=$ E (643); and the Angle C is common in both Triangles ADC, EBC; also the Angle ADE $=$ ABE, ADC $=$ EBC; and so the Triangles ADC and EBC are equiangular; therefore DC : CA : : BC : CE (657); whence $AC \times BC = EC \times DC$ (61).
Q. E. D.

663. *Corol.* Let EC remove into the Situation CF, touching the Circle in F; then will EC $=$ DC $=$ CF; and the Theorem will become $AC \times BC = CF^2$; whence in this Case, AC : CF : : CF : BC.

Theorem XXXII.

664. A Right-line AB, touching a Circle in the Point E, makes Right-angles with the Diameter CE.

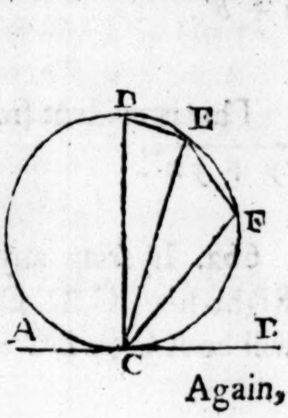


Demonstration.

If it be denied, let the Line FG make Right-angles therewith. Then the Side FE which subtends the Right-angle FGE, and which is equal to FD, is greater than the Side FG (636), *which is absurd.*

Theorem XXXIII.

665. If a Right-line AB touch a Circle in C, and from that Point be drawn a Right-line CE cutting the Circle, the Angles ECB, ECA, which it makes with Tangent-line, are equal to the Angles EDC, EFC, which are made in the alternate Segments of the Circle.



Demonstration.

Let CD be a Diameter or Perpendicular to AB, then the Angle CED is right (645); therefore the Angle D $+$ DCE $=$ L $=$ ECB $+$ DCE; therefore the Angle D $=$ ECB; the first Thing.
Q. E. D.

Again,

Again, because the Angle ECB + ECA = 2 L (626) = D + F (644), from both take away ECB = D (above), and there will remain ECA = F; the second Thing. *Q. E. D.*

THEOREM XXXIV.

666. If any Angle A of a Triangle BAC be bisected, the Right-line AD that bisects it shall divide the Base BC, so that BD : DC :: AB : AC.

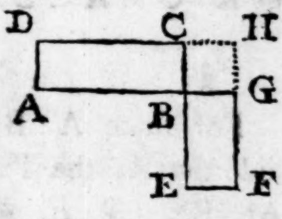
DEMONSTRATION.

Produce AB, and make AE = AC, and join CE; then because AE = AC, the Angle ACE = E (641) and ACE + E = BAC (632) and because BAD = DAC, therefore BAD = E; hence DA ∥ CE, (631) wherefore BA : AE (= AC) :: BD : BC (656). *Q. E. D.*

THEOREM XXXV.

667. Equal Parallelograms, having one Angle in each equal, *viz.* ABC = EBG, have the Sides about the equal Angles reciprocally proportional.

DEMONSTRATION.



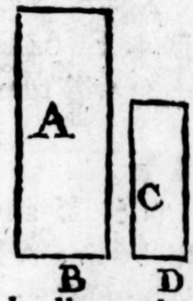
Let the Sides AB, BG, about the equal Angles make one Right-line, then shall EB, BC do the same, and produce FG, DC, till they meet. Then it is AB : BG :: BD : BH (655) :: BF : BH :: EB : BC, *Q. E. D.*

668. *Corol.* Hence the same Thing is evident in *Triangles* also, as being the Halves of the Parallelograms. From this Theorem appears the Reason of *resolving Equations into Analogies.* For let AB = a, BG = b, EB = c, BC = d. Then by Hypothesis $ad = cb$; and by the Theorem, $a : b :: c : d$.

THEOREM XXXVI.

669. Similar Parallelograms A × B and C × D are in a Duplicate Ratio, or as the Squares of their like Sides.

Demonstration.



Since $A : B :: C : D$, therefore $AD = BC$; multiply this Equation by the Rectangle BD, it produces $AB \times D^2 = CD \times B^2$; whence $AB : CD :: B^2 : D^2$, (668). *Q. E. D.*

670. *Corol.* Hence all ſimilar Triangles, and all regular Right-lined ſimilar Figures which are reſolvable into Triangles, are in the duplicate Ratio of their like or homologous Sides.

Theorem XXXVII.

671. If four Quantities are proportional, their Squares ſhall be ſo.

Demonstration.

Let the Quantities be A, B, C, D; and ſince $A : B :: C : D$, it is $AD = BC$; multiply this Equation by itſelf, it produces $A^2 B^2 = B^2 C^2$; wherefore $A^2 : B^2 :: C^2 : D^2$ (322). *Q. E. D.*

Theorem XXXVIII.

672. If three Quantities A, B, C, are proportional, it ſhall be $A : C :: A^2 : B^2$.

Demonstration.

For ſince $A : B :: B : C$, it is $AC = B^2$; multiply each Side by A, the Product is $A^2 C = B^2 A$; therefore $A : C :: A^2 : B^2$. *Q. E. D.*

673. *Scholium.* If the ſame Equation $AC = B^2$ be multiplied by B, it produces $ACB = BBB$, whence $B : C :: AB : B^2$. Conſequently $A : B :: AB : BB$. That is, *any two Numbers have the ſame Ratio, as the Rectangle made of them has to the Square of the Conſequent.*

Theorem XXXIX.

674. If four Quantities A, B, C, D, are proportional it will be $A : D :: A^2 : BC$.

De

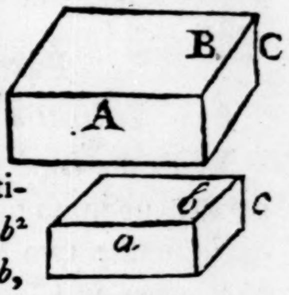
Demonstration.

Since $A : B :: C : D$, it is $AD = BC$; which multiplied on both Sides by A, gives $A^2 D = ABC$; therefore $A : D :: AA : BC$. *Q. E. D.*

Theorem XL.

675. Similar Solids are in the *Triplicate Ratio*, or as the Cubes of their like Sides.

Demonstration.



Let the three Dimensions of one Solid be A, B, C, and of the other a, b, c, respectively. Then since $A : a :: B : b$, it is $Ab = aB$. Again, $B : b :: C : c$, therefore $Bc = Cb$; let these Equations be multipled together, they make the Product $ACb^2 = B^2 ac$; multiply this on each Side by Bb, and it produces $ABCb^3 = B^3 abc$; whence $ABC : abc :: B^3 : b^3$. *Q. E. D.*

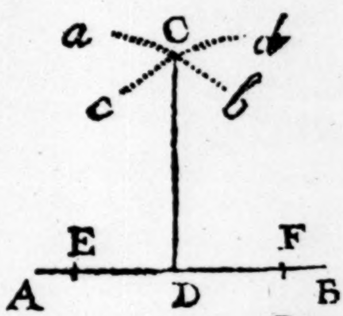
These forty Theorems will suffice for our Purpose at present, to which we shall add the following Problems necessary in *Practical Geometry*.

CHAP. III.

GEOMETRICAL PROBLEMS.

Problem I.

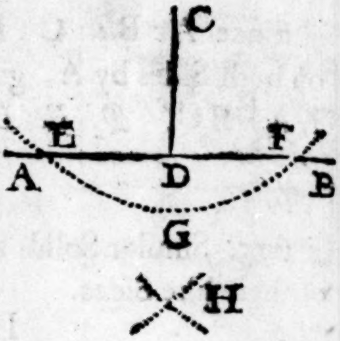
676. UPON *a given Right-line* AB *to erect a Perpendicular* CD, *in the Point* D.



Take on either Side the given Point D, the Distance $DE = DF$; with any opening of the Compasses, set one Foot in E, and F, and strike the Arches ab, and cd, cutting each other in C; join CD and it is the Perpendicular required.

Problem II.

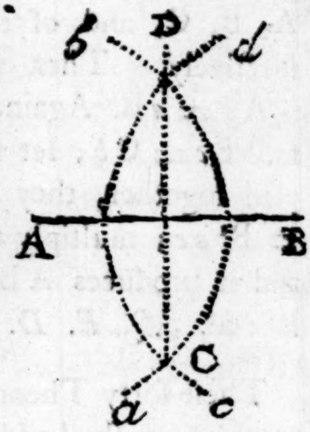
677. *Upon a Right-line* AB *from any given Point* C *above it, let fall the Perpendicular-line* CD.



On the given Point C, strike the Arch of a Circle EGF intersecting the Line AB in E and F; on the Points E and F, make the Intersection at H; lay a Ruler from C to H, and draw CD, it is the Perpendicular required.

Problem III.

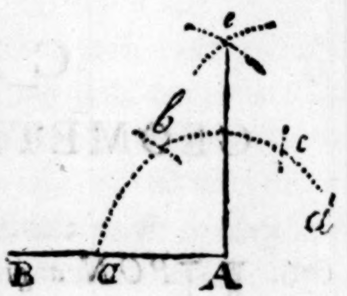
678. *To bisect a given Right-line* AB.



With the same Opening of the Compasses, upon the Extremities A and B, describe the two Arches *ab*, *cd*, cutting each other in C and D; thro' the Points C and D, draw the Right-line CD, and it will bisect the given Line AB, as required.

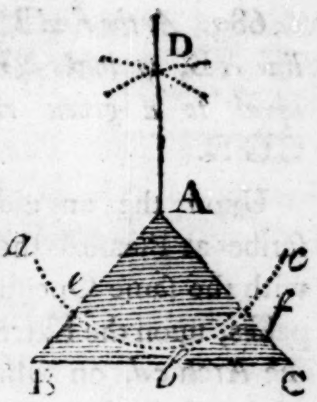
Problem IV.

679. *To erect a Perpendicular on the Extremity* A *of a given Right-line* AB.



On the Point A describe the Arch *ad*; and with the same Opening of the Compasses, from *a* make the Intersection *b*, and on *b*, the Intersection *c*; then from *b* and *c* make the Intersection *e*; and draw *e*A the Perpendicular required.

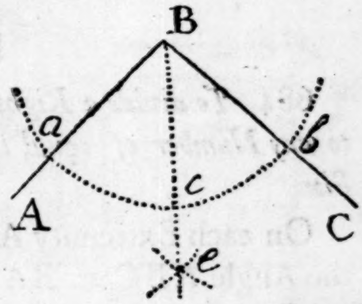
Problem V.



680. *On the Point* A *of a given Angle* BAC *to erect a Right-line* AD, *which shall incline neither to the Right-hand nor to the Left.*

On the angular Point A describe the Arch *abc*, cutting the Sides in *e* and *f*; on *e* and *f* make the Insection D, thro' which draw the Line AD, and it is done.

Problem VI.

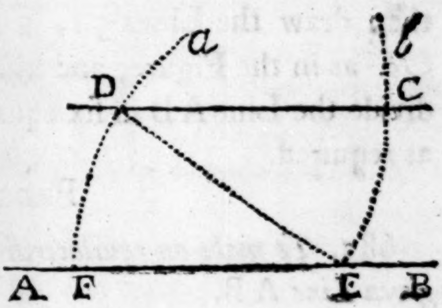


681. *To bisect a Right-lined Angle* ABC.

On the Point B describe the Arch *acb* cutting the Sides in *a* and *b*, on which Points makes the Intersection *e*, and draw the Line *e*B; and the Angle ABC is bisected thereby.

Problem VII.

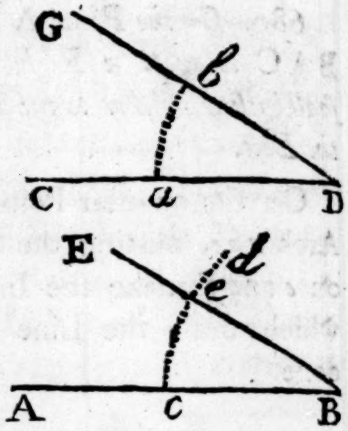
682. *Through a given Point* D, *to draw a Right-line* DC *parallel to a given Right-line* AB.



From the Point D draw at Pleasure the Oblique-line DE; on the Points D and E, describe the Arches E*b*, and F*a*; and make the Arch EC = FD; and then through the Points D, and C, draw the Line DC; and it will be parallel to AB.

Pro-

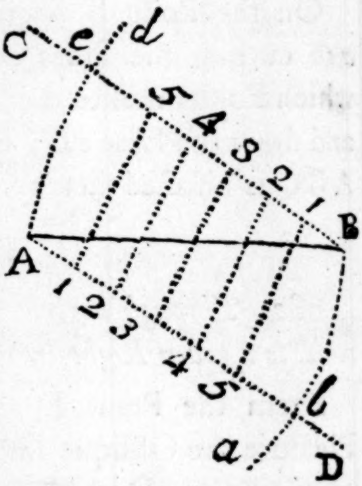
Problem VIII.



683. *At the End* B *of a given Right-line* A B, *to make a right-lined Angle equal to a given right-lined Angle* C G D.

Upon the angular Point D describe at Pleasure the Arch *a b*; and with the same Opening of the Compasses, upon the Extremity B describe the Arch *c d*, on which make *c e* = *a b*; and thro' the Points B, *e*, draw the Line E B, and it will make the Angle A B E = C D G.

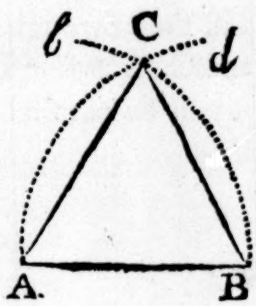
Problem IX.



684. *To divide a Right-line* A B *into any Number of equal Parts, suppose Six.*

On each Extremity A and B make the Angle A B C = B A D; then on the Line B C with any small Opening of the Compasses make the Five equal Distances, at 1, 2, 3, 4, 5, and also do the same on the Line A D, then draw the Lines 5 1, 2 4, 3 3, &c. as in the Figure, and they shall divide the Line A B in six equal Parts as required.

Problem X.

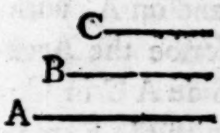


685. *To make an equilateral Triangle upon a given Line* A B.

Upon the Extent A with the Distance A B, describe the Arch B *b*; and on the Extent B with the same Distance describe the Arch A *d*, cutting the other in C, join A C, B C, and it is done.

Problem XI.

686. *To make a Triangle whose three Sides shall be equal to three given Right-lines* A, B, C.

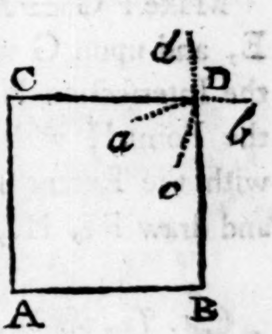


Make DE = A, and on D with the Extent of the Line B describe the Arch *ab*; and on E, with the Extent of the Line C, describe the Arch *cd*, cutting the former in F; join DF, EF, and it is done.

Problem XII.

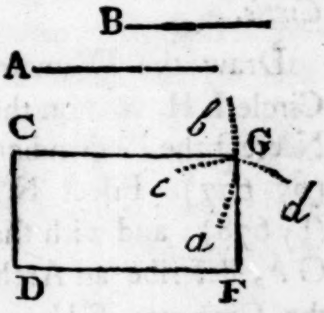
687. *To make a Square upon a given Right-line* AB.

On the Extent A erect the Perpendicular AC = AB (679); with the Extent AB on the Points B, C, describe the Arches *ab*, *cd*, intersecting in D; draw CD, BD, and it is done.



Problem XIII.

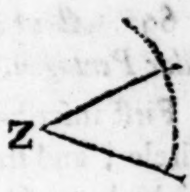
688. *To make a Parallelogram, whose two unequal Sides are equal to two given Right-lines* A, B.



Make DE = A; and on the Extent D, erect the Perpendicular DC = B; then on the Point C, with the Extent of A, describe the Arch *ab*; and on F, with the Extent of B, describe the Arch *cd*, intersecting the former in G; draw CG, FG, and it is done.

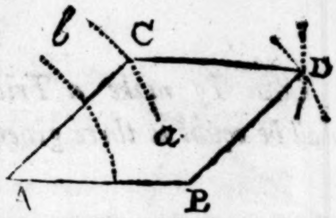
Problem XIV.

689. *To make a Rhombus* on the Right-line AB whose acute Angle shall be equal to a given right-lined Angle Z.



Make

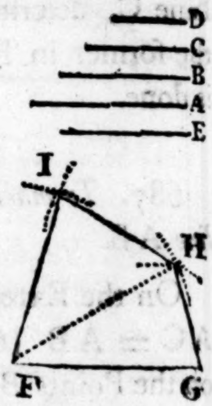
Make the Angle A = Z (by 683) and on A, with the Extent of A B, describe the Arch *a b* and it will cut the Side A C in C; on the Points C and B with the Extent A B, describe the Arches making the Intersection D; join C D, B D, and it is done.



N. B. *In like Manner you make a given Rhomboides.*

PROBLEM XV.

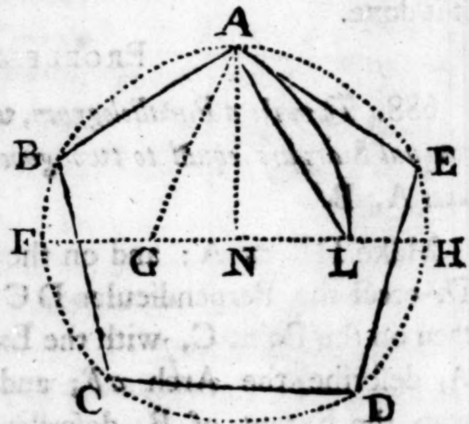
690. *To make a Trapezium whose Sides shall be equal to four given Right-lines* A, B, C, D, *and whose Diagonal is equal to a Right-line* E.



Make F G = A; upon F with the Extent of E, and upon G with the Extent of D, make the Intersection H, and draw G H. Then on the Point H with the Extent of C, and on F with the Extent of B, make the Intersection I; and draw F I, H I, and it is done.

PROBLEM XVI.

691. *To inscribe an equilateral and equiangular Pentagon in a Circle.*



Draw the Diameter of the Circle F H, and on the Center N erect the Perpendicular A N (by 677). Bisect N F in G (by 678); and with the Radius G A, describe an Arch cutting the Diameter F H in L, and draw A L; it shall be the Side of the Pentagon required; * *viz.* A L = A B = B C = C D = D E = E A.

PROBLEM XVII.

692. *About a given Circle to describe a regular Pentagon.*

First inscribe a regular Pentagon in the Circle; and from the Center C draw the Right-lines C A, C B, C D, &c. to the several Angles thereof; and to these Lines draw Perpendiculars *a b*, *a e*, *e d*, *d c*, *c b*, and it is done.

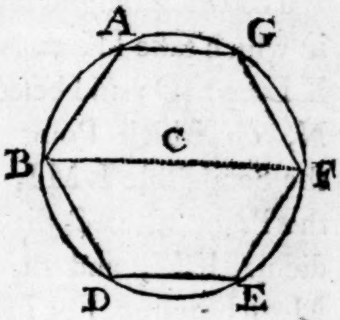
PRO-

* See Dr. *Barrow*'s Scholium to Prop. X. Book XIII. of *Euclid*.

Problem XVIII.

693. *To inscribe a regular Hexagon in a Circle.*

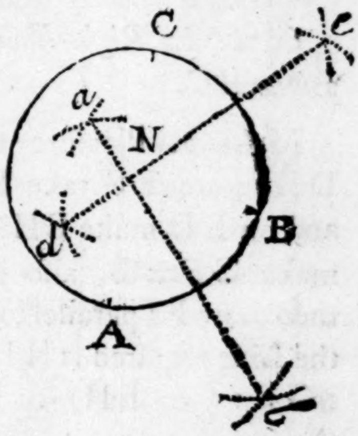
Draw the Diameter B F, the Half of which, B C, set round upon the Circle gives the several Points, A, B, C, D, E, F, G; which joined, constitute the Hexagon required.



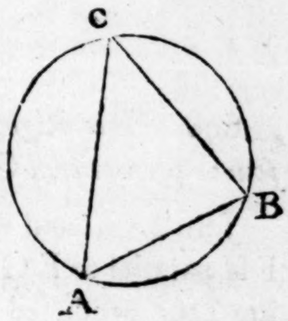
Problem XIX.

694. *To find the Center of a Circle.*

Take any three Points A, B, C, in the Circumference, and upon A and B, with the same Opening of the Compasses make the Intersections *a* and *b*, and draw the Line *a b*. On the Points B and C, make the Intersections *d*, *e*, and draw *d e*, it will intersect *a b* in N, the Center of the Circle that was to be found.



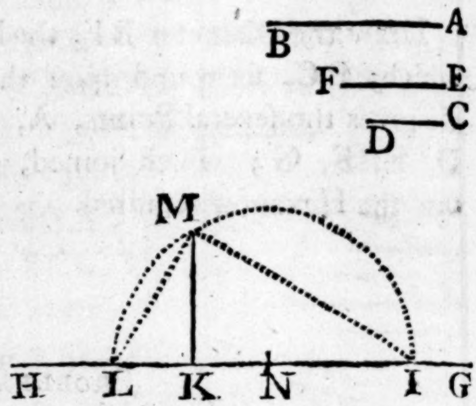
695. N. B. *Hence it is easy to draw a Circle through any three Points given*, A, B, C. *Also any Part of a circular Arch* A B C *may be compleated into a Circle.* And thus also *may a Circle be circumscribed about the given Triangle* A, B, C.



Problem XX.

696. *To find a mean proportional* EF *between two given Right-lines* AB, CD.

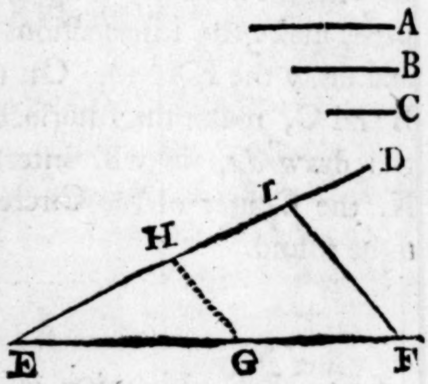
Draw any Right-line GH, in which take IK = AB, and KL = CD; and beſect IL in N, on which Point deſcribe the Semicircle LMI, then on the Point K erect the Perpendicular KM, and that is the Mean required. For IK : KM :: KM : KL (by 645 and 660.)



Problem XXI.

697. *Two Right-lines* A *and* B, *being given to find a third proportional* C.

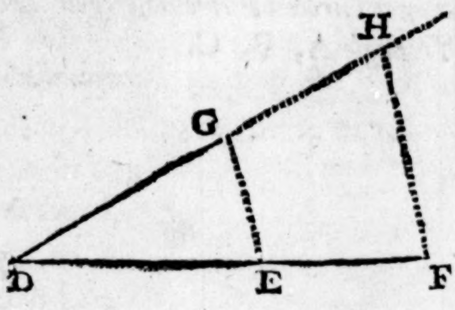
Make at Pleaſure the Angle DEF; on EF take EG = A, and on ED make EH = B; and make GF = B, alſo join HG; then draw FI parallel to GH; and the Line required is HI. For EG : GF : (= EH) :: EH : HI (by 656.)



Problem XXII.

698. *Three Right-lines being given* DE, EF, DG, *to find a fourth proportional* GH.

Join EG, and thro' F draw FH parallel to EG, and meeting DG produced in H; then is GH the Line required. For DE : EF :: DG : GH.



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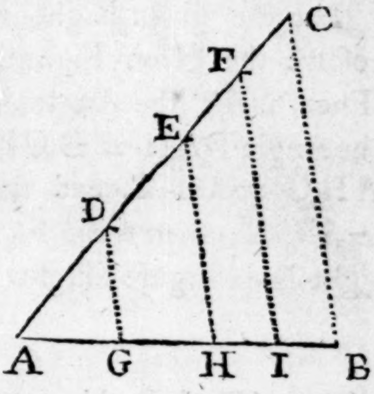
PROBLEM XXIII.

699. *To divide a Right-line* AB *as another given Right-line* AC *is divided.*

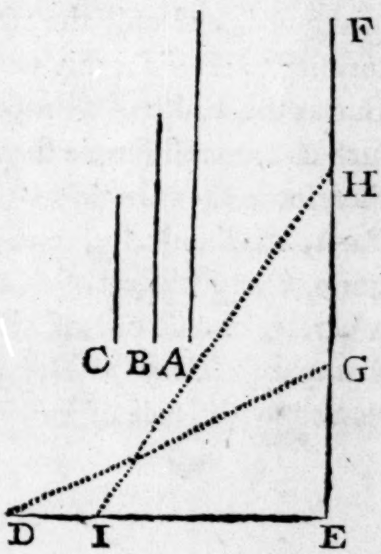
Join the Extremities of each Line CB, and parallel to CB draw EI, EH, DG through the ſeveral Points of Diviſion in the Line AC, and they ſhall divide the Line AB as AC is divided.



PROBLEM XXIV.

700. *To make a Square equal to any Number of Squares given.*

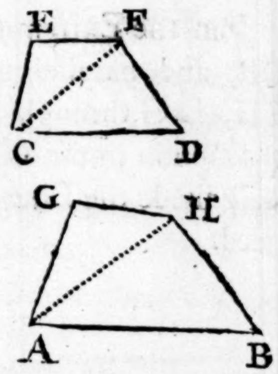
Let the Sides of three given Squares be the Right-lines A, B, C, and make the Right-angle DEF; and take $ED = A$, and $EG = C$, and join DG; then is $DG^2 = DE^2 + GE^2$ (646.) Again, make $EH = DG$, and $EI = B$; and join IH. Then is $IH^2 = IE^2 + EH^2 = EI^2 + DE^2 + GE^2 = A^2 + B^2 + C^2$. And thus you proceed for any other Number.



701. What is here done with regard to Squares, holds good for all ſimilar plain Figures deſcribed on the Sides of the Right-angled Triangle, becauſe they have all the *Ratio of the Squares* of their like Sides (by 669, 670); and how to make one Figure *ſimilar* to another is ſhewn in the following

PROBLEM XXV.

702. *On a given Right-line to make a right-lined Figure* AGHB *ſimilar and alike ſituated to a given right-lined Figure* CEFD.



Let the given Right-line be AB, and reſolve the given Figure into Triangles. Then make the Angle ABH = D, and the Angle BAH = DCF, and the Angle AHG = CFE, and the Angle HAG = FCE; then is the Figure AGHB the right-lined Figure ſought.

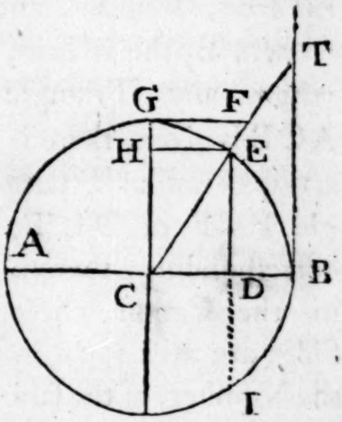
703. Theſe Problems are in general ſelf-evident to every one who underſtands the preceeding Theorems, ſo that a formal Demonſtration would have been Tautology; the Rationale of ſome is aſſigned, and only the 16th will give the Reader the Trouble of turning to *Euclid*; this we ſhall probably hereafter demonſtrate. But as the Truth of a Propoſition may be ſhewn different Ways, ſuch as are moſt ſimple ſhould firſt be propoſed to Learners; and therefore, as a Circle may be divided into any Number of equal Parts, mechanically, to as great Exactneſs as any Praxis can require, ſo a *fifth* Part of that Number will aſſign an Arch of ſuch a Circle, the Chord of which will be the Side of an inſcribed *Pentagon*. Thus 72 Degrees is the 5th Part of 360, and is ſubtended by the Side of a *Pentagon*.

INSTITUTIONS OF Plain TRIGONOMETRY.

Of RIGHT-ANGLED TRIANGLES.

704. BEFORE we can treat of the *Method of measuring Heights* and *Distances*, and other *Trigonometrical* Operations, so frequent in the *practical Part of the Mathematics*, it is necessary in the first Place, to explain the *Theory*, that the Grounds and Reasons thereof may appear; and which we shall make the Subject of this Chapter.

705. Let AIBG be a Circle, whose Diameter is AB, and Center C. On the Point B erect the Perpendicular TB, which shall touch the Circle in B, and is therefore called a TANGENT Line; and draw CT cutting the Circle in E, and the Tangent in T, this is called the SECANT of the Arch BE. Through E draw EI ∥ TB, then is EI called the CHORD of the Segment EBI, or EAI;



and Half thereof ED is called the SINE of the Arch EB or EGA. The Semidiameter CE in this Case is called the RADIUS.

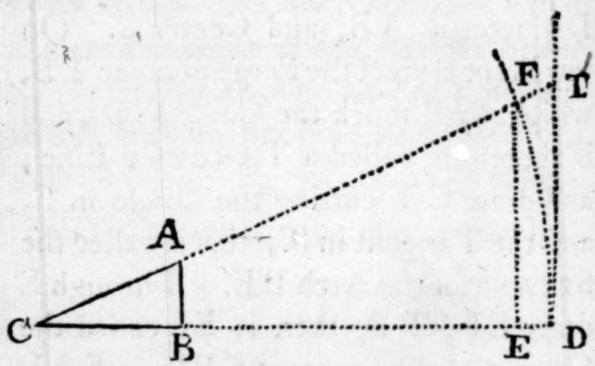
706. As every Circle is supposed to be divided into 360 equal Parts, called DEGREES, so one fourth Part, *viz.* 90 Degrees, is the Measure of the *quadrantal Arch* GB, or the Right-angle GCB. Hence whatever Number of Degrees are contained in the Arch BE, the Arch EG contains the Remainder or Complement to 90; and is therefore called the Complement of EB to a Right-angle or Quadrant BG. Hence the Tangent GF, Sine EH, and Secant CF, are said to be the *Sine*, *Tangent*, *and Se-*

Secant of the Complement EG, or in ſhort, the *Co-ſine*, *Co-tangent*, and *Co-ſecant* of the Arch BE.

707. Therefore in any Right-angled Triangle CDE, if the Side CE (ſubtending the Right-angle) which is called the *Hypothenuſe*, be made *Radius*; then is the perpendicular Leg, or *Cathetus* ED, the *Sine* of the Angle ECB at the Baſe; and the Baſe CD is the Co-ſine thereof; for it is equal to HE, the Sine of the complemental Arch GE (706.)

708. In any Right-angled Triangle CBT, if either of the Sides be made *Radius*, as CB; then the other Side BT is the Tangent, and the Hypothenuſe CT is the Secant of the Angle at C.

709. In Calculations of this Kind, the Sides of the Triangle are eſtimated in Parts, of which the Radius contains 10000000; and by the Induſtry of our Predeceſſors, the Sides of a Triangle, whether *Sines*, *Tangents*, or *Secants*, are computed in this Meaſure of equal Parts for every Magnitude of the Angle BCG, that is, for every Degree and Minute of the Quadrant BG; and theſe Numbers diſpoſed in Tables, are called the *Canon* of *natural Sines*, *Tangents*, and *Secants*.



710. By this Means, when any Triangle ACB is given, there is always a ſimilar Triangle ECF or DCT, correſponding thereto in the Canon whoſe Sides are all known in the Numbers of the ſaid Canon; with theſe the Sides of the given Triangle are compared, and by that Means, thoſe which are unknown become known.

711. Thus, ſuppoſe in the given Triangle ACB, you have found the Angle ACB $= 25° : 30'$, and the Side CB $= 356$, to find the Side AC and AB. Then the Triangle in the Table, equiangular to the given Triangle, is FCE, if AC be made Radius. For in this Caſe, the tabular Radius CF $=$ 10000000, the Sine of $25° \; 30' =$ FCD, is FE $= 4305111$; and the Co-ſine, or Baſe CE $= 9025853$. Hence, becauſe of

ſimilar

similar Triangles FEC, ABC, we have the following Analogies (657.)

$$\text{As}\begin{cases} EC & : BC \\ 9025853 & : 356 \end{cases} \begin{cases} :: FE & : AB \\ :: 4305111 & : 169{,}8. \end{cases}$$
$$\begin{cases} :: FC & : AC \\ :: 10000000 & : 394{,}4. \end{cases}$$

712. But if the Base CB be made Radius, then we have the tabular Triangle DCT; in which the Base CD = 10000000; the tabular *Tangent* DT = 4769755, and *Secant* CT = 11079285, by which we find AB and AC, as follows.

$$\text{As}\begin{cases} CD & : CB \\ 10000000 & : 356 \end{cases} \begin{cases} :: DT & : AB \\ :: 4769755 & : 169{,}8. \end{cases}$$
$$\begin{cases} :: CT & : AC \\ :: 11079285 & : 394{,}4. \end{cases}$$

713. But since in these natural Numbers, the Operation is very laborious and tedious; their *Logarithms* have been made use of, and digested into Tables which are called the *logarithmical or artificial Canon of Sines, Tangents and Secants.* By these the irksome Task of *multiplying and dividing* large Numbers is prevented, being answered by the Addition and Subtraction of their Logarithms. (See Inst. 153 — 155.)

714. Thus the above Operations by Logarithms are very easy, as follows:

		Logarithms.
As the Co-sine of the Angle C = 25° : 30′, *viz.*	EC = 9025853 =	9,9554882
is to the Base —— —— ——	BC = 356 =	2,5514500
So is the Sine of C = 25° : 30′, *viz.* ——	EF = 4305111 =	9,6339844
		12,1854344
To the Side or Cathetus —— ——	AB = 169,8 =	2,2299462

715. Or by Tangents thus,

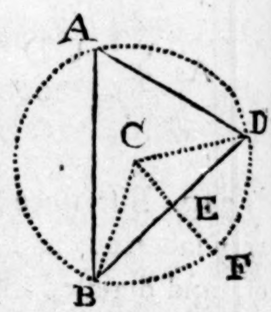
As Radius —— ——	CD = 10000000 =	10,0000000
to the Base —— ——	BC = 356 =	2,5514500
So is the Tab. Log. Tangent of 25° : 30′ =	DT = 4769755 =	9,6784961
to the Side —— ——	AB = 169,8 =	2,2299461

Here the Reader will observe the Side AB is found much easier by the Tangent than by the Sine, because of the Radius, which

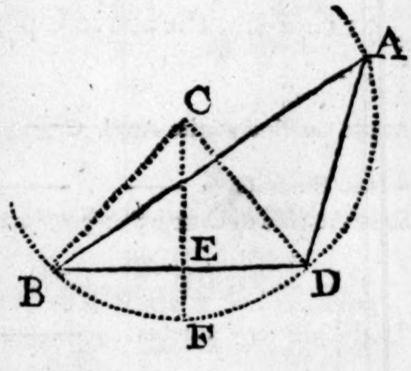
which gives no Trouble in the Work. Therefore in Calculations we ſhould always chuſe that Analogy in which the Radius makes one Term.

716. In a Right-angled Triangle, if one Angle be known, the other is known alſo, as being the Complement to 20 Degrees (706.) Therefore if any one Side and one Angle be given, the other two Sides may be found; or if two Sides be given, the other Side and the Angles may be found in the Manner above exemplified, of which we ſhall have Inſtances enough hereafter.

OBLIQUE ANGLED TRIANGLES.



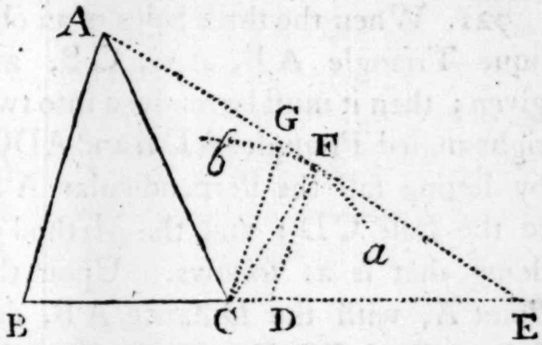
717. About the Acute angled Triangle ADB, circumſcribe a Circle (695,) from whoſe Center C let fall the Perpendicular CE to the Side DB, which it will biſect in E, and the Arch BD in F. Join CD and CB, then is the Angle DCB = 2 DAB (642) = 2 FCB; therefore the Angle DAB = FCB; but BE is the Sine of the Angle FCB, and therefore alſo of the Angle DAB. Thus $\frac{1}{2}$ AB is the Sine of the Angle ADB; and $\frac{1}{2}$ AD the Sine of ABD.



718. The ſame Demonſtration holds for any Obtuſe angled Triangle as is evident in the annexed Figure; and therefore ſince in every oblique Triangle, the Half-ſides are as the Sines of the oppoſite Angles, the whole Sides will have the ſame Ratios. And hence, if two Sides AB, AD, and an oppoſite Angle B be given, the other Parts of the Triangle will be thereby found. For as AD : Sine of ABD : : AB : Sine of A, which is therefore known. Then as the Sine of B : AD : : Sine of A : BD; and ſo the whole Triangle is known.

719. In

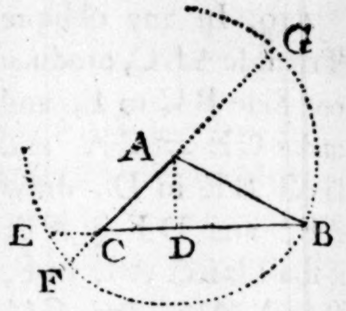
719. In any oblique Triangle ABC, produce one Side BC to E, and make CE = CA, and bisect BE in D; draw AE, and DF ∥ AB, it shall bisect AE in F, (699.) Also, draw CG ∥ AB.



On the Center C, with the Radius CF, describe the Arch *a*F*b*; then because the Angle ACE = B + BAC (= 2 ACF) by (632.) and the Angle GCA = A; if therefore from the half Sum $\frac{A+B}{2}$ = ACF you take the lesser Angle A = ACG, there will remain ½ their Difference $\frac{B-A}{2}$ = GCF. Also, if from the ½ Sum of the Sides $\frac{BC+AC}{2}$ = BD, you take the lesser Side BC, there will remain ½ their Difference $\frac{AC-BC}{2}$ = CD (221, 228.) Now AF is the Tangent of the Angle ACF, and GF is the Tangent of GCF (705.) and it is BD : CD :: AF : GF (656.) *Therefore, as ½ the Sum of the Sides* $\frac{AC+BC}{2}$ *is to half their Difference* $\frac{AC-BC}{2}$, *so is the Tangent of half the Sum of the opposite Angles* $\frac{A+B}{2}$ *to the Tangent of half their Difference* $\frac{B-A}{2}$.

720. By this Theorem when any two Sides AC, BC, of a Triangle are given, with the included Angle C; the other Side and Angles become known. For since the Sum of the two Angles A + B = 180 — C (633.) it is a known Quantity; and having found ½ their Difference by the Analogy (719.) by adding it to ½ the Sum, it gives the greater Angle B, (221.) and by subtracting it from ½ the Sum you have the lesser Angle A. And having found all the Angles, the Side AB is known by (718).

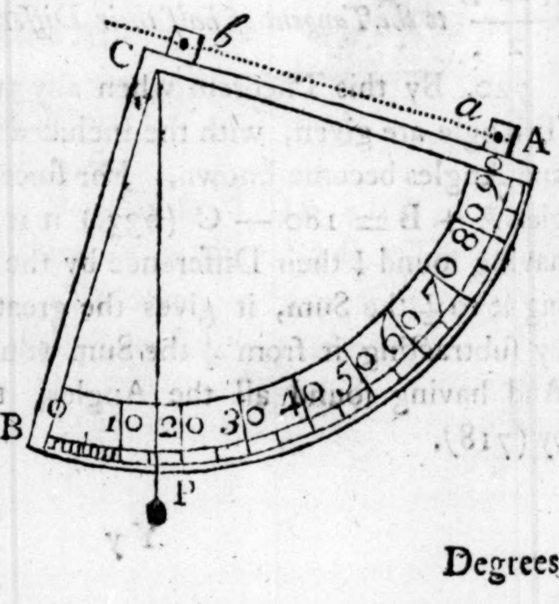
721. When the three Sides of an oblique Triangle AB, AC, CB, are given; then it must be resolved into two right angled Triangles ADB and ADC, by letting fall the Perpendicular AD to the Base CD; and the Method of doing that is as follows. Upon the Point A, with the Distance AB, describe a Circle EBG. Continue out the Side AC, both Ways, to cut the Circle in F and G, and the Side BC to cut it in E; then is $GC = AC + BA$, and $DE = DB$; therefore, $EC = BD - CD$; thus also $FC = AB - AC$. But it is $FC \times CG = EC + CB$ (658.) whence $CG : CB :: CE : CF$; that is, $AC + AB : CB :: AB - AC : EC$; wherefore $\frac{1}{2}CB + \frac{1}{2}CE = DB$, and $\frac{1}{2}CB - \frac{1}{2}CE = CD$ (221.) Then in each right-angled Triangle ABD, ADC, there are two Sides given, to find the Angles, by (710, 711.) And these are all the Particulars and Varieties in the Theory of plain Trigonometry, which will suffice for our present Purpose; hereafter we may treat more largely on this Subject.



CHAP. XI.

Of measuring HEIGHTS, DEPTHS, *and* DISTANCES.

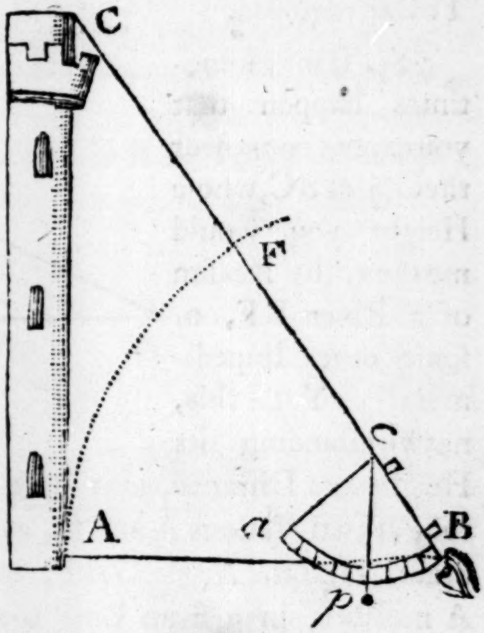
722. THIS Kind of Mensuration depends upon the Use of the QUADRANT in taking Angles. The Manner of doing which is therefore first to be explained. The *Quadrant* is so called from its being a *fourth Part of a Circle*, as ACB, whose Limb AB is accurately divided into 90



Degrees,

Degrees, and when the *Quadrant* is large, each Degree is subdivided into 10 equal Parts, each containing six Minutes; in the Center C is fixed a small Pin, from which there hangs a Plumb-line, CP, whose Plummet P keeps it tightly stretch'd over the Degrees on the Limb when in Use. Upon the Side A C, are two thin Plates of Brass, *a b*, with small Perforations, or Holes, exactly in a right Line *a b*, parallel to the Side of the *Quadrant*, A C.

723. To illustrate the Use of the Quadrant in taking Angles, let C A be a very high Tower; and suppose a Person, B, taking the Angle of its Altitude A B C; this he does by holding up the Quadrant *a c* B, and moving it about 'till, thro' the Sights, he perceives the Top of the Tower C; then will the Plumb-line *c p* cut the Limb in that Number of Degrees and Minutes which are equal to the Angle of Altitude ABC, or so as to make the Arch *a p* of the same Number of Degrees as are contained in A F, described on the Radius A B. For supposing A B parallel to the Horizon, *c p* parallel thereto (as being perpendicular to AB,) and B C the visual Ray by which the Point C appears thro' the Sights. Then the Angle B A C $=$ (B + C $=$) *a c* B $=$ *a c p* $+$ *p c* B; but *p c* B $=$ A C B (631.) therefore *a c p* $=$ C B A. *Q.E.D.*

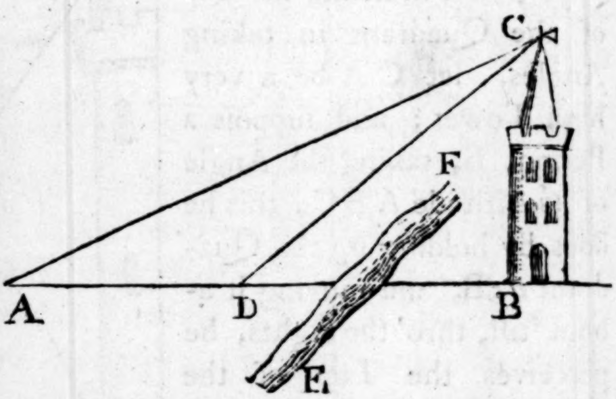


724. Hence the Height of the Tower D C is easily found thus; measure the Distance E D upon the Ground with a Chain or Pole, and suppose it be found 150 Yards. Then with your Quadrant at E take the Angle of Altitude A B C, which suppose you find 52° 30′. Then in the right angled Triangle ABC, there is known the Base A B $=$ D E $=$ 150, and the Angle at Base A B C $=$ 52° 30′, to find the Cathetus A C. Which you do by the following Analogy (715.)

As Radius	———	———	10.
to the Base,	———	AB = 150 =	2.1760913
So is the Tangent of		ABC = 52° 30′ =	10.1150195
to the Height	———	AC = 195,5 =	2.2911108

Let the Height of the Observer B = 6 Feet, or 2 Yards; then AC + AD (or B) = DC = $197\frac{1}{2}$ Yards, the Height of the Tower required.

725. It may sometimes happen that you cannot come near the Object BC, whose Height you would measure, by Reason of a River EF, or some other Impediment. Yet this, notwithstanding its



Height and Distance, may both be measured by taking its Altitude at two Stations A and D, as follows. At A, with the Quadrant, take the Angle CAB, which let be 24° 45′; then from A measure, in a strait Line towards the Object, the Distance AD = 100 Rods; and at D take the Angle CDB, which suppose = 38°. Then is the Angle ADC = 180 — 38 = 142′. So that in the obtuse Triangle ADC, there is known the Side AD, and the adjacent Angles A and D, to find the Side DC, which is done thus. The Angle ACD, CDB — CAB = 38° — 24° 45′ = 13° 15′. Then say, (718.)

As the Sine of the Angle		ACD = 13° 15′ =	9.360215
is to the Side	———	AD = 100 =	2.000000
So is the Sine of the Angle		DAC = 24° 45′ =	9.621861
to the Length of the Side		DC = 182,6 =	2.261646

726. Then in the right angled Triangle DBC, there is given the Hypothenuse DC, and the Angle at Base, to find the Sides CB, and DB. Therefore say,

As Radius ——— ——— 10.
to the Side — ——— BC = 182,6 = 2.261646
So is the Sine of the Angle CDB = 38° 00′ = 9.789342

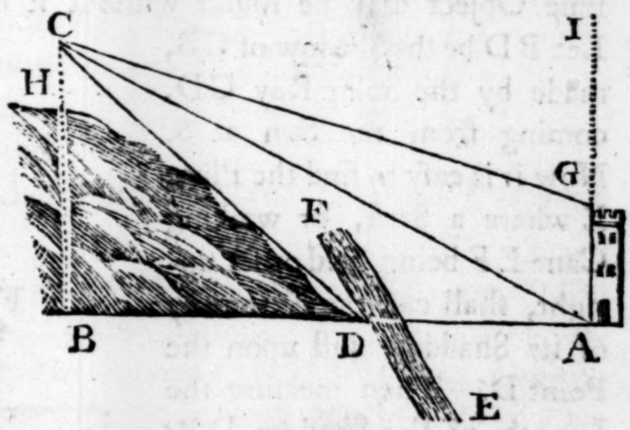
to the Height of the Spire CB = 112,46 = 2.050988

727. Then to find the Diſtance DB,

Say, as Radius ——— ——— 10.
to the Side ——— BC = 182,6 = 2.261646
So is the Sine of the Angle DCB = 52° 00′ = 9.896532

to the Diſtance ——— DB = 143,9 = 2.158178

728. In the ſame Manner, if you were ſtationed on the Top of an high Hill, as at H, and at the Foot of the Hill a River EF, or ſome other Obſtacle prevented your Acceſs to the diſtant Object at A;



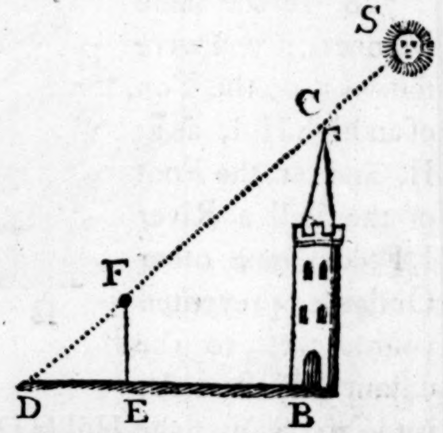
yet if the Side of the Hill HD was not too irregular, you might eaſily find the Height of the Hill, and the Diſtance and Height of the Object AG. Thus, let the ſlant Height, or Slope of the Hill HD be meaſured, and ſuppoſe it is found 182,6 Rods. Then will this differ inſenſibly from the Hypothenuſe CD of the right angled Triangle BCD; with the Quadrant you take either of the Angles BCD, or BDC; thus let BDC = 38°, then will BCD = 52°. And having the Hypothenuſe CD = 182,6 and Angle BDC = 38°. You find the perpendicular Height of the Hill (or Eye at C) to be CB = 112,46; and the Baſe of the Hill BD = 143.9 (by 727.)

729. Having thus got the Height of the Hill, take with your Quadrant the Angle BCA, which you will find to be 65° 15′; and becauſe the Angle BDC = 38°, the Angle CDA = 142°, and the Angle DCA = 65° 15′ — 52° = 13° 15′. Therefore in the obtuſe Triangle DCA, there is given all the Angles, and the Side CD, by which Means the Side DA will be found 100 Rods

Rods (725.) Which is the Distance of the Object from the Foot of the Hill required.

730. Moreover if it be required to measure the Height of a distant Object A G; then you next find the Side C A of the Triangle A D C; the Angle B C G being taken with the Quadrant, is equal to the Angle C G I; whence the Angle C G A is known. Also the Angles C A G = A C B = 65° 15′; hence all the Angles, and the Side A C in the Triangle A G C are known. Then say, as the Sine of C G A (or I G C, (705.) is to the Side A C :: the Sine of A C G to the Side A G = the Height of the Object required.

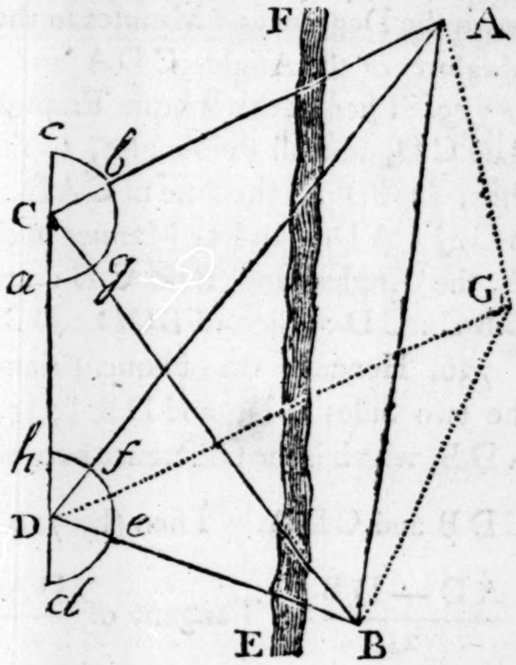
731. If a Quadrant be not at Hand, the Height of any accessible Object may be found without it by the Shadow, thus. Let B D be the Shadow of C B, made by the Solar Ray C D, coming from the Sun at S. Now it is easy to find the Place E where a Staff, or walking Cane E F being held or set upright, shall cast the Extremity of its Shaddow just upon the Point D. Then measure the Length of the Shadow D B,



and D E, and as the Height of the Staff E F is known, you say (by 657.) As the Shadow of the Staff D E, is to that of the Object D B, so is the Height of the Staff E F to the Height of the Object B C, which, therefore is known. There are other Ways of doing this without a Quadrant also, but none more easy, and so ready as this upon an Exigency.

N. B. As the measuring the *Depth* of a Valley, or other Place, depends on the same Principles and Construction of Figures, I apprehend it will not be necessary to insist on that Article here, as there can be no Difficulty attending it, if Institution 728 be at all understood; since the *Height of the Hill* is the same Thing with the *Depth of the Vale.*

732. To



732. To meaſure the Diſtance between two or more inacceſſible Objects A, B, you muſt be provided with a Semi-circle, with a moveable Index carrying *two Sights*, and a Spirit-Level for fixing it truely horizontal; a *Plane-Table* alſo will anſwer the End. And a *Theodolite* much better than any other Inſtrument. Some Inſtrument of this Sort muſt be had, and it matters not much for the Form; we will therefore ſuppoſe it the common graduated *Semi-circle.*

733. In order, then, to meaſure the Diſtance A, B, on the other Side of the River E F, two Stations muſt be determined, as C and D, in which to fix the Semi-circle for taking Angles; let the Diſtance C D be nicely meaſured, for the Baſe of your future Triangles. Then over the Point C erect your Three-legged Staff bearing the Semi-circle, and adjuſting it to a true horizontal Poſition, move the Index to the Beginning of the Semi-circle, or Line C *a*, and move it about 'till thro' the Sights you perceive an upright Pole, *&c.* fixed before-hand in the Point D. Here let the Semi-circle reſt; and then move the Index 'till thro' the Sights at C *g* you ſee the Object B, and write down the Number of Degrees and Minutes contained in the Arch of the Semi-circle *ag*, for that will be the Meaſure of the Angle DCB. Again, move the Index to C *b*, where you obſerve the Object A, and write down the Degrees and Minutes in the Arch *a b*, which is the Meaſure of the Angle D C A.

734. This done, remove your Semi-circle to D, and there ſetting it up, with its Centre over D, lay the Index to the Side D *h*, and move the Semi-circle about 'till thro' the Sights you obſerve the upright Pole, *&c.* in C. In this Poſition ſcrew the Semi-circle faſt, and move the Index to D *f* and D *e*, and write down

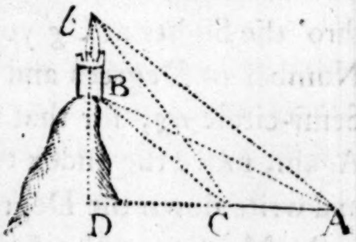
down the Degrees and Minutes in the Arches *hf* and *he*, for the Measures of the Angles CDA and CDB.

735. Then in the oblique Triangle CAD, there is given the Base CD, and all the Angles, to find the Side AD; which is thus, (718.) As the Sine of CAD : CD : : Sine of ACD (or AC*c*) : AD. In like Manner in the Triangle CBD you have all the Angles and Base CD; and therefore, as the Sine of DBC : CD :: Sine of BCD : DB.

736. Hence in the oblique Triangle ADB, there is known the two Sides AD, and DB (735.) and the included Angle ADB, which is the Difference between the known Angles (734.) CDB and CDA. Then (by 719.) you say, as $\frac{AD + DB}{2}$: $\frac{AD - DB}{2}$ :: Tangent of $\frac{DAB + DBA}{2}$: Tangent of $\frac{DAB - DBA}{2}$. Then, as you have $\frac{1}{2}$ the Sum, and $\frac{1}{2}$ the Difference, of those Angles, the Angles themselves are known (by 720.)

737. Lastly, In the Triangle ADB, all the Angles, and two Sides AD and DB are known to find the third Side AB, thus, (718.) as the Sine DAB : DB :: Sine of ADB : AB = the Distance between the two Objects A and B as required. In the same Manner the Distance between A and G, or B and G may be found; and thus the most useful Doctrine of plain Triangles extends to *Surveying*, *Fortification*, *Navigation*, and every other Art where right-lined Figures are concerned.

738. The same Method is taken for measuring the Height of an Object B*b*, on the Top of an inaccessible Mountain BD, as is evident by a bare Inspection of the Figure, the Construction here being the same as the Foregoing, and therefore the Particulars need not be repeated.



CHAP. XII.

INSTITUTIONS OF CONIC GEOMETRY.

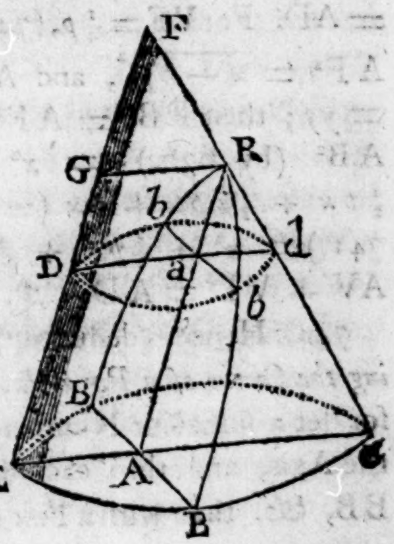
CHAP. XII.

The Chief Properties *of the* CONIC SECTIONS *demonstrated.*

739. BEFORE we can proceed to the Menſuration of *Superficies* and *Solids*, it will be neceſſary to premiſe the Principles of CONIC GEOMETRY, and the Method of FLUXIONS; as by theſe we ſhall come to the Rationale of thoſe uſeful Practices, ſooner and eaſier than by any other Way. Nor is the Reader to wonder, if he be told, that he can never guage a Veſſel, or meaſure a Piece of Timber, *according to Art*, 'till he has firſt learned *Conics* and *Fluxions*, or obtained the ſame Ideas ſome other Way, which he will find much more difficult, laborious, and irkſome. I have been at the Pains to try every Way myſelf, and therefore am able to point out, to the young *Tyro*, that which is pleaſanteſt and beſt.

Of the PARABOLA.

740. If a Cone be cut thro' by a Plane parallel to one of its Sides E F, the Figure of the Section A B R B is called a PARABOLA; and its Property is thus inveſtigated. E B *e* B is a Circle, E *e* a Diameter, and B B (the Baſe of the Parabola) a right Line at right Angles therewith. A R is the Axis of the Parabola; draw G R parallel to E *e*. Put F G $= a$, E A $= b$, A R $= x$, and A B $= y$. Then, becauſe of ſim. Triangles,



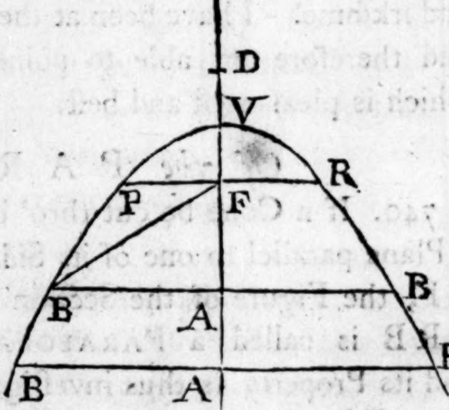
FGR

FGR and R A e, it is F G : G R (= E A) :: A R : A e, that is, $a : b :: x : \frac{bx}{a} = \mathrm{A}e$. But $\mathrm{EA} \times \mathrm{A}e = \mathrm{AB}^2$ (by 658.) That is $\frac{b^2 x}{a} = yy$. Now put $\frac{b^2}{a} = p$, then $px = yy$. After the same Manner it is shewn, that, (drawing D d ‖ E e, and bb ‖ B B, and putting R a = x, and ab = y) it is $p\mathrm{x} = \mathrm{yy}$, therefore $p = \frac{yy}{x} = \frac{\mathrm{yy}}{\mathrm{x}}$; whence $\mathrm{y}^2 : y^2 :: \mathrm{x} : x$. Or, R a : R A :: a b² : A B², every where.

741. The Part of the Axis R a, or R A, is called the *Abscissa*, the Lines a b, A B *Ordinates*, or Lines ordinately applied to the Axis. The invariable Line p is called the *Parameter*, or *Latus Rectum*, being a third Proportional to any Abscissa and its Ordinate; for since $px = yy$, we have $x : y :: y : p$.

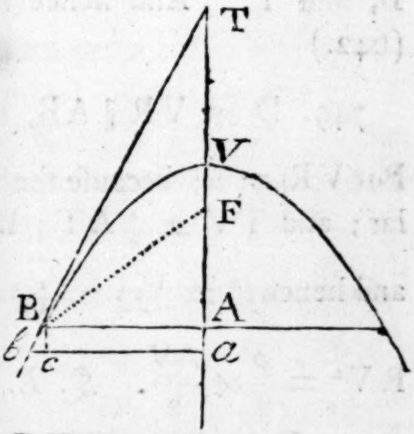
742. Since $y = \mathrm{AB} = \frac{1}{2}\mathrm{BB}$, it is $yy = \frac{1}{4}\mathrm{BB}^2 = xp$, or $4px = \mathrm{BB} \times \mathrm{BB}$. Now among all the Ordinates B B, some one will be equal to the Parameter p; in that particular Case we have $4px = pp$, or $4x = p$, and so $x = \frac{1}{4}p$; that Point of the Axis, where this Parameter-ordinate intersects it, is called the Focus *of the Parabola*. As the Point F, when the Ordinate $\mathrm{PR} = 4\,\mathrm{VF}$, the *Abscissa*. See the following Figure.

743. In the Axis continued, take V D = V F, the Distance of the Focus from the Vertex, and join F B; then it will be F B = A V + V F = AD. For $\mathrm{VF} = \frac{1}{4}p$, (742.) $\mathrm{AF}^2 = \overline{x - \frac{1}{4}p}^2$, and $\mathrm{AB}^2 = yy$; then $\mathrm{FB}^2 = \mathrm{AF}^2 + \mathrm{AB}^2$ (by 636.) $= x^2 - \frac{1}{2}px + \frac{1}{16}pp + px$ (= yy, 741.) $= x^2 + \frac{1}{2}px + \frac{1}{16}pp$; therefore $\mathrm{FB} = x + \frac{1}{4}p = \mathrm{AV} + \mathrm{VD} = \mathrm{AD}$. *Q.E.D.*



744. Hence is deduced the best and easiest Method of *describing the Curve of a Parabola*, whose Focal Distance V F is given, for let a sufficient Number of Points A, A, *&c.* be taken in the Axis, and thro' each draw perpendicular and parallel Lines B B, *&c.* then with a Pair of Compasses take the Distance A D, and

and with one Foot in the Focus F, interſect each Parallel reſpectively, in B, B; then with an even Hand draw a Curve through all thoſe Points, and it ſhall be the *Parabola* propoſed.



745. If any Right-line touch the Parabola as in B, the ſame ſhall meet the Axis produced in T, ſo as to make AV = TV. For let B*b* be an indefinitely ſmall Part of the Curve coincident nearly with the Tangent TB. Draw *ba* ∥ BA, and B*c* ∥ V*a*; and put B*c* $= n$, $bc = m$, and TV $= a$. Then V$a = x + n$, and $ab = y + m$; and in the Triangles *b*B*c*, BTA, we have $m : n :: y : x + a$, therefore $n \dfrac{y}{m} = x + a$; but $p \times \mathrm{V}a = \overline{ab}^2$, that is, $px + pn = y^2 + 2ym + m^2$, but $px = yy$; therefore $pn + yy = y^2 + 2ym$ (for m^2 is too ſmall to be regarded) hence $pn = 2yn$ and $n = \dfrac{2ym}{p} = \dfrac{xm + ma}{y}$, or $\dfrac{2y}{p} = \dfrac{x + a}{y}$, that is, $2yy = px + pa$, and $p = \dfrac{2yy}{x + a} = \dfrac{yy}{x}$, conſequently $2x = x + a$, therefore $x = a$, or AV = TV.

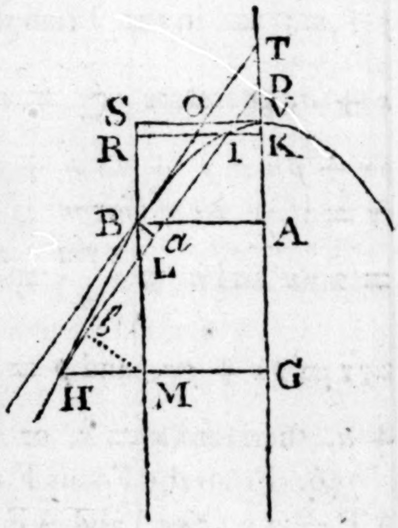
746. From the Focus F draw FB, it ſhall be FB = FT, for AT $= 2x$ (745.) and AF $= x - \frac{1}{4}p$; therefore FT = AT — AF $= 2x - x + \frac{1}{4}p$ = FB (by 743).

747. To the Tangent TB, let CB be Perpendicular in the Point B, and meet the Axis in C; then it will be AC $= \frac{1}{2}p$. For let AC $= b$, then (by 660.) it is AT : AB :: AB : AC, that is, $2x : y :: y : b = \dfrac{yy}{2x} = \frac{1}{2}p$, (by 741.) = AC.

748. Moreover it is, $FB = FC$; for $AF = x - \frac{1}{4}p$, and $AC = \frac{1}{2}p$ (by 747.) therefore $FC = x - \frac{1}{4}p + \frac{1}{2}p = x + \frac{1}{4}p = BF$ (746.) $= FT$.

Coroll. Hence F is the Center of a Circle paſſing through C, B, and T. And hence alſo the Angle $CTB = \frac{1}{2}CFB$. (642.)

749. Draw $VR \parallel AB$, then it ſhall be $VR^2 = \frac{p \times AV}{4}$. Put $VR = b$; becauſe the Triangles TVR, TAB, are ſimilar; and $TV = \frac{1}{2}AT$; therefore $RV = \frac{1}{2}AB$, or $b = \frac{1}{2}y$; and hence $b^2 = \frac{1}{4}yy = \frac{1}{4}px$ (741.) $= \frac{p}{2} \times \frac{x}{2}$; therefore $b^2 = RV^2 = \frac{p}{2} \times \frac{AV}{2}$. *Q. E. D.*



750. Draw $DH \parallel TB$ a Tangent, cutting the Curve in I and H; and from the Point B, draw $BM \parallel VG$, this is called *a Diameter to the Point* B; and it ſhall biſect the Part IH within the Curve. For through the Points V, I, H, draw VS, KR, GH, parallel to AB, and meeting BM produced in S, R, M. Then becauſe $VO = \frac{1}{2}AB = \frac{1}{2}VS$, and $TV = SB$, (745.) therefore the Triangle $TVO = SBO$, and $\triangle ABT = \square AS$. But $\triangle ABT$ (or AS) : KID :: ($AB^2 : KI^2$ (670.) :: AV : KV (740.) AS : KS (655). Therefore $\triangle KID = \square KS$. Again, $\triangle ABT$ ($= \square AS$) : $\triangle GHD$ :: ($AB^2 : GH^2$ :: AV : GV) $\square AS : \square GS$; therefore $\triangle GHD = \square GS$. But $GHD - KID = GS - KS$; therefore the Trapezium $GHIK = \square GR$. From each of which, take the common Figure GMLIK, and there will remain the Triangle $ILR = LMH$; and as they are ſimilar, we have $LI = LH$, whence the Line IH is biſected in L by the Diameter BM, to which therefore the Line HI is an *Ordinate*.

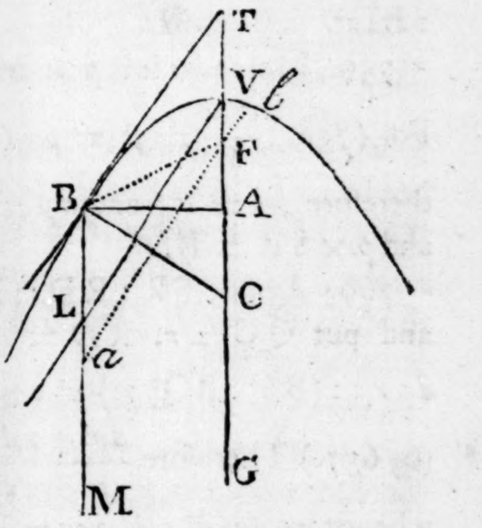
751. The Parallelogram $LBTD = \triangle MLH$; for $ABT = AS$, whence the Trapezium GMBT ($= GB + ABT = GB + AS = GS$) $=$ (750.) GHD; from which take the com-

common Figure HMLD there remains LBTD = △ LMH. *Q. E. D.*

752. From B let fall the Perpendicular Ba; and from M the Perpendicular Mb. Then because $BT \times Ba = \frac{LH \times Mb}{2}$, or $2\,BT \times Ba = LH \times Mb$; therefore $2\,BT : LH :: Mb : Ba :: LM : LB$, by similar Triangles. Consequently $2\,BT \times LB = LM \times LH$. Let $BS : BO :: 2\,BT : P$. Then shall this fourth proportional P be the *Parameter* belonging to the Diameter B M.

753. Then, the Rectangle of this Parameter and any *Abscissa* of its Diameter is equal to the Square of Ordinate of that Abscissa. Or $P \times BL = LH^2$ or LI^2. For since $BS : BO :: LM : LH$, by similar Triangles; we have $2\,BT : P :: LM : LH$; whence $P = \frac{2\,BT \times LH}{LM}$, and (by 752.) it is $LB = \frac{LM \times LH}{2\,BT}$. Therefore $P \times LB = \frac{2\,BT \times LH}{LM} \times \frac{LM \times LH}{2\,BT} = LH^2$. *Q. E. D.*

754. The Parameter P is equal to $p + 4\,AV$. For draw V L parallel to the Tangent T B, this will be an Ordinate to the Diameter B M (by 750). Then because BM ∥ VG, it is $TV = BL = AV$ (745.) $= x$; and (by 753.) $P \times BL$, or $P \times x = VL^2 = BT^2 = AB^2 + AT^2 = y^2 + 4\,x^2$, therefore $Px = 4\,x^2 + px$ (741.); whence $P = 4\,x + p = p + 4\,AV$. *Q. E. D.*



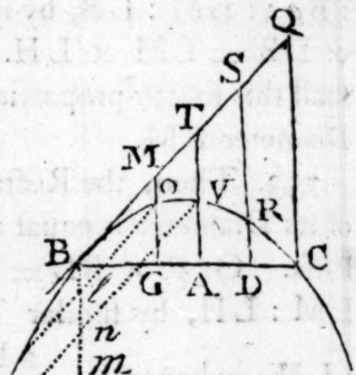
755. Let F be the Focus; and then $BF = \frac{1}{4}P$. For $P = p + 4\,AV$ (754.) and $p = 4\,FV$ (742). Therefore $P = 4\,AV + 4\,FV$, and $\frac{1}{4}P = AV + FV = FT = BF$ (by 746.) *Q. E. D.*

756. If an Ordinate ab to any Diameter B M pass thro' the Focus F, then it shall be $\frac{1}{4}P^2 = ab^2$, or $\frac{1}{2}P = ab$. For because of Parallels, $aB = FT = BF$ (755.) $= \frac{1}{4}P$. Also

P ×

$P \times ab = (753.)\ \overline{ab^2}$; therefore $\frac{1}{4}P \times P = \overline{ab^2}$; whence $\frac{1}{2}P = ab$. *Q. E. D.*

757. Let CB be perpendicular to the Tangent in B; then is $CT = \frac{1}{2}P$. For $BF = \frac{1}{4}P$ (755.) $= CF = FT$ (748). Consequently $CT = CF + FT = \frac{1}{4}P + \frac{1}{4}P = \frac{1}{2}P$. *Q. E. D.*



758. If a Right-line QB touch the Parabola in B, and from the Points M, S, in the Tangent, the Right-lines MG, SD are drawn parallel to the Axis VA, cutting the Ordinate BC in G and D; then it will be $MO : SR :: BG^2 : BD^2$. Put $MO = b$, $SR = d$, $BG = c$, $BD = a$; also let $AV\ (= VT) = x$; then drawing $Bm \parallel VA$, and Ol, $Vn \parallel BQ$; it is $x : b :: (Bn : Bl :: nV^2 : Ol^2 ::)\ BT^2 : BM^2 :: y^2\ (= AB^2) : c^2\ (= BG^2)$ (by 657 and 671). For the same Reason $x : d :: y^2 : a^2$. Therefore (by 652.) we have $b : d :: c^2 : a^2$, or $MO : SR :: BG^2 : BD^2$. *Q. E. D.*

759. Moreover it is $p \times MO = BG^2$, or $p \times SR = BD^2$. For (758.) $\frac{c^2 x}{b} = y^2 = px$ (by 741). Also $\frac{a^2 x}{d} = y^2 = px$; therefore $pb = c^2$, and $dp = a^2$; that is, $p \times MO = BG^2$, and $p \times SR = BD^2$.

760. Again, $SR : RD :: BD : DC$; for draw $QC \parallel VA$, and put $QC = r$, $RD = s$, and $DC = q$, $R = p$. Then $d : r :: (BS^2 : BQ^2 ::)\ a^2 : \overline{a+q}|^2$, and $r : a + q :: d + s : a$. (by 657.) Therefore $\frac{da^2 + 2daq + dq^2}{a^2} = r = \frac{ad + as + qd + qs}{a}$; whence we get $a^2 s + aqs = daq + dq^2$, and dividing by $a + q$, it is $as = dq$; hence $d : s :: a : q$, or $SR : RD :: BD : DC$. *Q. E. D.*

761. Again, $AV : AB^2 :: RD : BD \times DC$. For $x : d :: y^2 : a^2$ (758.) and $d : s :: a : q$ (760.) therefore $\frac{xa^2}{y^2} = d = \frac{sa}{q}$, hence $xqa = y^2 s$; then $x : y^2 :: s : aq$, or $AV : AB^2 :: RD : BD \times DC$. *Q. E. D.*

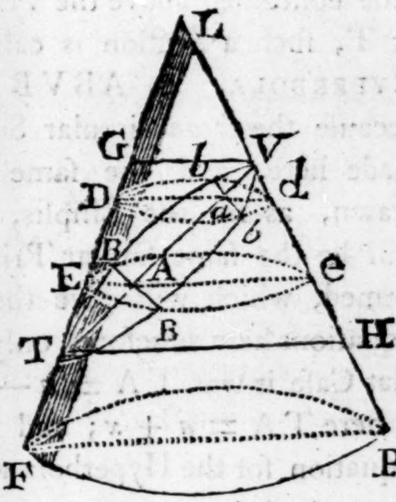
Coroll.

Coroll. Hence also (AV : AB² : :) OG : BG × GC : : RD : BD × DC.

762. Because the $y^2 = px$ (741.) and $y^2 = \frac{aqx}{s}$ (761.) therefore $px = \frac{aqx}{s}$, and $ps = aq$; that is, $p \times$ RD = BD × DC. Thus also $p \times$ OG = BG × GC. *Q. E. D.*

Of the Ellipsis.

763. If a Cone FLP be cut by a Plane which passes through both its Sides, that Section VBTBV is called an Ellipsis; and its general Property is thus investigated. Suppose EBeB a circular Section parallel to the Base, and cutting the Ellipse in the Right-line BB; then shall Ee the Diameter of the Circle intersect TV the longest (called the *Transverse*) *Diameter* of the Ellipse in the Point A, bisecting the Ordinate BB; then draw GV and TH parallel to FP; and put GV = d, TH = c, VT = a, AB = y, VA = x. Then because of similar Triangles VAe, VTH, we have VT : TH : : VA : Ae = $\frac{cx}{a}$. Also because of the similar Triangles TGV, TEA, it is, TV : GV : : TA : AE = $\frac{\overline{a-x} \times d}{a}$ = $\frac{ad - xd}{a}$. But because of the Circle EBeB, we have EA × Ae = AB × AB = (658.) AB²; that is, $\frac{cxad - cxxd}{aa} = yy$.

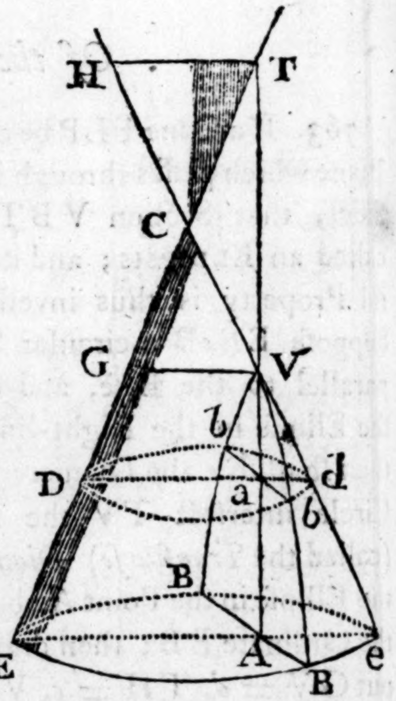


764. Let it be made $a : d : : c : p = \frac{cd}{a}$; then substituting p for its Value in the above Equation, it becomes $\frac{pax - pxx}{a} = yy$. And if another circular Section DbdbD be made cutting the Elliptic one in bb, then ab = y, and Va = x, it will appear

pear by the ſame Reaſoning, that $\frac{pax - pxx}{a} = yy$. Hence $ax - xx : yy :: a : p :: ax - xx : yy$. Conſequently it will be every where $a : p :: \overline{a - x} \times x : y^2$.

Of the HYPERBOLA.

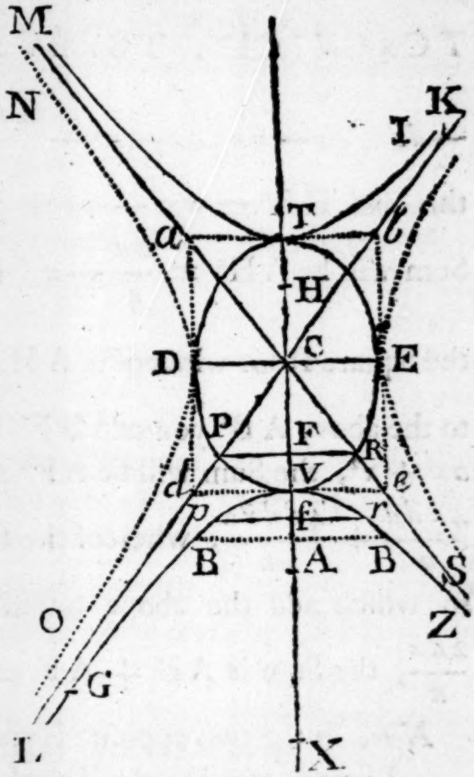
765. If a Cone E C *e* be cut by a Plane thro' the Baſe, and the oppoſite Side continued above the Vertex as at T, ſuch a Section is called an HYPERBOLA, as ABVB; and becauſe the ſame circular Sections made here, and the ſame Lines drawn, as for the Ellipſis, there will be the ſame ſimilar Triangles formed, which will give the ſame Equations here as before, only as in that Caſe it was TA $= a - x$, it is here TA $= a + x$; and ſo the Equation for the Hyperbola will be this $\frac{pax + pxx}{a} = y^2$, or in Analogy, $a : p :: \overline{a + x} \times x : yy$. Hence the Analogy $a : p :: \overline{a \mp x} \times x : yy$ will ſerve for both *Ellipſis* and *Hyperbola*, at the ſame Time; uſing the Sign — for the former, and + for the latter in all Caſes.



766. The greateſt Ordinate DE in the Ellipſe is called the *Conjugate Diameter*. Let the Half thereof of ED $= b$; in this Caſe $x =$ VC $= \frac{1}{2} a$, and $y = \frac{1}{2} b$. Whence the Analogy for the Ellipſe becomes $a : p :: \overline{a - \frac{1}{2} a} \times \frac{1}{2} a : \frac{1}{2} b \times \frac{1}{2} b$; that is, $a : p :: \frac{1}{4} aa : \frac{1}{4} bb$, and ſo $ap = bb$; whence $a : b :: b : p$. So the *Parameter of the Ellipſe is a third Proportional to the Tranſverſe and Conjugate Diameter.*

767. Since $a : p :: aa : bb$ (766.) $:: \overline{a - x} \times x : yy$ (764.) we have $AB^2 : TA \times VA :: DE^2 : TV^2$, or $\frac{bb}{aa} \times \overline{ax - xx} = yy$.

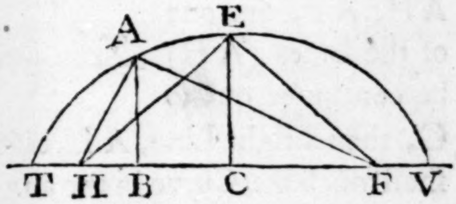
768. Becauſe, in the Focus, $y = \frac{1}{2}p$, or $yy = \frac{1}{4}pp$, we ſhall, in that Caſe, have $\frac{1}{4}p = \frac{ax - xx}{a}$, or $\frac{1}{4}ap = ax - x^2$; and changing all the Signs it is $x^2 - ax = -\frac{1}{4}ap$, and compleating the Square it is $x^2 - ax + \frac{1}{4}aa = \frac{1}{4}aa - \frac{1}{4}ap$; extract the Root, and $x - \frac{1}{2}a = \sqrt{\frac{1}{4}aa - \frac{1}{4}ap} = \frac{1}{2}\sqrt{aa - ap}$. Let F, H, be the two Focus's; then $CF = CH = \frac{1}{2}a - x = \frac{1}{2}\sqrt{aa - ap}$, or $\sqrt{aa - ap} = 2FC = FH$; whence $aa - ap = FH^2$, and ſo TV $(= a) : FH :: FH : TV - p$ $(= a - p.)$



769. Two Right Lines drawn from any Point in the Curve of the Ellipſis to the two Foci, are, together, equal to the Tranſverſe Diameter.

CASE I.

Let E, the End of the Conj. Diameter, be the given Point; and draw HE, FE; then is $HE + FE = TV$. For $CE^2 + CH^2 = HE^2$; but $CE^2 = \frac{ap}{4}$, and $CH^2 = \overline{\frac{1}{2}a - x}|^2 = \frac{aa - ap}{4}$; whence $\frac{ap}{4} + \frac{aa - ap}{4} = \frac{aa}{4} = EH^2$; therefore $EH = \frac{1}{2}a = EF$; conſequently $EH + EF = \frac{1}{2}a + \frac{1}{2}a = a = TV$. (766, 768.)



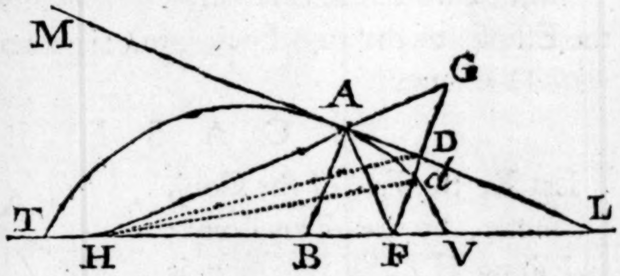
CASE II.

Let A be a Point taken any where in the Curve, and draw HA, FA; and the Ordinate AB. Put $TB = x$, $HC = d$; then $TH = \frac{1}{2}a - d$, $HB = x - \frac{1}{2}a + d$, and becauſe $HE = TC = \frac{1}{2}a$, therefore $CE^2 = \frac{aa}{4} - dd$. But

$TC \times CV : CE^2 :: TB \times BV : AB^2$ (767.) that is $\frac{aa}{4} : \frac{aa}{4} - dd :: ax - xx : ax - xx - \frac{4xdd}{a} + \frac{4ddxx}{aa} = AB^2$; to this add $HB^2 = xx - ax + \frac{1}{4}aa + 2dx - ad + dd$, the Sum will be $AH^2 = \frac{aa}{4} - ad + dd + 2dx - \frac{4xdd}{a} + \frac{4ddxx}{aa}$, the Square Root whereof is $AH = \frac{1}{2}a - d + \frac{2dx}{a}$. Alſo, if to the above AB^2 you add $BF^2 = d^2 + da + \frac{1}{4}aa - 2dx - ax + x^2$, the Sum will be $AF^2 = d^2 + da + \frac{1}{4}aa - 2dx - \frac{4xdd}{a} + \frac{4ddxx}{aa}$, whereof the Root is $d + \frac{1}{2}a - \frac{2dx}{a} = AF$; to which add the above found Equation $AH = \frac{1}{2}a - d + \frac{2dx}{a}$, the Sum is $AH + AF = \frac{1}{2}a + \frac{1}{2}a = TV$. *Q.E.D.*

Note, in the two oppoſite Hyperbolas, the Difference of theſe Lines is equal to the Tranſverſe TV.

770. If to any Point A of the Curve right Lines be drawn from the *Foci*, *viz.* AH, AF, and one of the Lines (AH) be continued out to



G, then a right Line, AL, biſecting the external Angle FAG, ſhall touch the Curve in the angular Point A. For make $AG = AF$, then (becauſe the Angle $GAL = FAL$) if you take any Point, D, in the Line AI, we have $GD = DF$ (by Inſt. 681.) draw HD, and Hd, then $HD + DG$ is greater than HA $(+ AG =) + AF = Hd + dF = TV$. (769.) Therefore the Point D is without the Curve, and will be ſo 'till it coincides with the Point A, conſequently AL is a Tangent to the Curve in that Point.

771. The two Lines AH, AF, drawn from the Point of Contact A to the two *Foci*, make equal Angles with the Tangent MAL. For the Angle FAL (= GAL (by 770.) = MAH (630.)

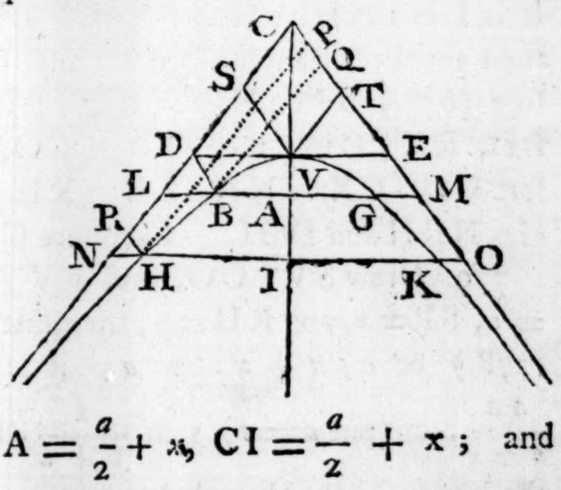
772. Let

772. Let A B be perpendicular to the Tangent in the Point of Contact A, then shall the Angle H A B = F A B. For the Angle M A B = L A B, by Supposition; from which take the Angle M A H = L A F (771.) there will remain the Angle H A B = F A B, (200.)

773. The same Construction remaining; we have H A : A F :: H B : B F. For in the Triangle H A F, the Angle H A B = B A F (772.) Therefore (by 666.) HA : AF :: HB : BF.

774. Since one and the same Line T V is the common Axis both to the *Ellipsis* and to the *Hyperbola*; (see Fig. to Inst. 768.) if we also take one common conjugate Diameter D E to both; and about the Ellipsis describe the Rectangle *abed*. Then, since p is the same in the *Ellipsis* and *Hyperbola*; we have C f = C d, or the Distance of the Focus f of the Hyperbola from the Center C, equal to half the Diagonal of the Rectangle *abed*. For putting V f = x, we shall find (by the Process of 768.) $\sqrt{\frac{1}{4}aa + \frac{1}{4}ap} = \frac{1}{2}a + x$ = C f; but $ap = bb$ (by 766.) therefore $\sqrt{\frac{1}{4}aa + \frac{1}{4}bb} =$ C f $= \sqrt{CV^2 + Vd^2} = Cd$ (636.) *Q. E. D.*

775. The Diagonals of this rectangular Parallelogram, produced both Way indefinitely, make those right Lines which are called the Asymptotes, whose Property it is *to approach nearer and nearer to the Curve of the Hyperbola inscribed between them, but will never coincide therewith.* Thus C S, C L are the *Asymptotes* of the Hyperbola G V Z; C K, C M of the opposite Hyperbola. Also C M, C L are Asymptotes to the Hyperbola O D N; and C Z, C K of the Opposite one.



776. That the Asymptotes constantly approaches the Curve but can never meet it, is thus shewn. Let C V = $\frac{1}{2}a$, V D = $\frac{1}{2}b = \sqrt{\frac{ap}{4}}$ or $VD^2 = \frac{ap}{4}$; V A = x, V I = x; then C A = $\frac{a}{2} + x$, C I = $\frac{a}{2} + x$; and

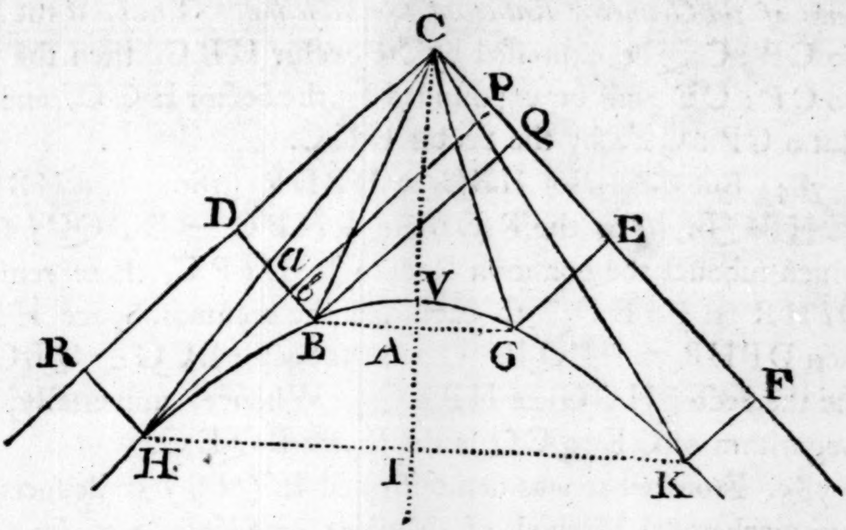
$CV : VD :: CA : AL = \sqrt{\frac{ap}{4}} + \sqrt{\frac{ap}{4}} \times \frac{2x}{a}$; and therefore $AL^2 = \frac{ap}{4} + px + \frac{pxx}{a} = \frac{ap}{4} + yy$ (765.) $= \frac{ap}{4} + AB^2$; therefore $AL^2 - AB^2 = \frac{ap}{4} = VD^2$. And by reasoning in the same Manner we prove $IN^2 - IH^2 = VD^2$. Therefore, since the Square of the Ordinate to the Curve is every where less than the Square of the same Ordinate to the Asymptote, by the constant Difference of $\frac{ap}{4}$ or VD^2, 'tis evident the Asymptote can never touch the Curve. *Q. E. D.*

777. In the Hyperbola, it will be every where $VD \times VE = LB \times BM = NH \times HO$. For $VD \times VE = VD^2 = \frac{1}{4}ap = AL^2 - AB^2 = IN^2 - IH^2$ (776.) and $AL + AB = BM$, and $AL - AB = LB$; therefore $\overline{AL + AB} \times \overline{AL - AB} = AL^2 - AB^2 = LB \times BM = \frac{1}{4}ap$; therefore $LB = \frac{\frac{1}{4}ap}{BM}$. In the same Manner we prove $NH = \frac{\frac{1}{4}ap}{HO}$; wherefore, since HO is greater than BM, 'tis evident NH is less than LB; so that the Curve is nearer the Asymptote at H than at B. And thus it will ever be nearer, but never meet it (by 776.)

778. Since $LB \times BM = (\frac{1}{4}ap =) NH \times HO$ (777.) It is $LB : NH :: HO : BM$. Draw HP, BQ $\parallel$ CN; then are the Triangles QBM and PHO similar, and therefore (657.) $BM : HO :: BQ : PH$ ($:: NH : LB$.) Draw BD, RH $\parallel$ CO; then $QB = CD$, and $PH = CR$; therefore $CD : CR :: NH : LB :: RH : DB$, by similar Triangles NRH and DBL. Therefore $CD \times DB = CR \times RH$.

779. Draw SV $\parallel$ CO, and TV $\parallel$ CN; and put $SV = CS = a$, $SR = x$, and $RH = y$; then, since $CS : CR :: RH : SV$, (778.) or $a : a + x :: y : a$; it is $aa = ay + yx$, or $y = \frac{aa}{a + x}$, or (put $a = 1$) $y = \frac{1}{1 + x}$, for the Equation expressing the Nature of the Hyperbola between the Curve and the Asymptotes.

780. If



780. If the Asymptotes make a Right Angle RCF, then the Hyperbola is said to be *Rectangular*. From the Point C draw CH, CB, CG, CK. The $\square$ CRHP = CDBQ (778.) Therefore the Triangle RHC = DBC; subtract from each the common Triangle CD*a*, there remains the Trapezium RDAH = $\triangle$ C*a*B; to each of these add the common Space BAH, the Sums are DBHR = HBC, the Hyperbolic Sector.

781. Because of Parallels, the Hyperbolic Trapezium GEFK = DBHR, and so the Sector CKG = CHB: and supposing the Ordinate BG to move parallelly, the Sector CVH = CVK.

782. The Line CF is divided proportionally in P Q E F, that is, CP : CQ :: CE : CF. For (778.) RH (= CP) : DB (= CQ) :: CD (= CE) : CR (= CF). Wherefore, since the Hyperbolic Spaces HRCP, HRCQB, HRCEG, HRCEK, are in *Arithmetical Progression* (being as 1, 2, 3, 4, *&c.*) and CP, CQ, CE, CF, are in *Geometrical Progression*, 'tis evident the Former are *Logarithms* of the Latter; that is, the *Hyperbolic Spaces are* Logarithms *of the Asymptotic Abscissæ*, (by 130, 131.)

783. Because when CP : CQ :: CE : CF, the Sector BCH = GCK; therefore when it is made CP : CQ :: CQ : CE, then the Sector BCH = BCG. Consequently, when CP, CQ, CE, CF, are proportional, the Sector HBC = BCG = GCH. And so the Sectors HBC, HGA, HKC, are in *Arithmetical Progression*, and therefore are *the Logarithms or Exponents*

nents of the Geometric Ratios of the Abſciſſæ. Thus, if the Ratio CP : CQ be expreſſed by the Sector HBC, then the Ratio CP : CE will be expounded by the Sector HGC, and the Ratio CP : CF, by the Sector HKC.

784. But the Sector HBC = HBDR (780.) and HBDR = HPQB. For the Rectangle RHPC = DBQC; from which ſubduct the common Rectangle D*b*PC, there remains D*b*HR = P*b*BQ; to each add the common Space H*b*B; then DBHR = HPQB. Thus the Sector BCG = QBGE; and the Sector HCG = HPEG. Whence, univerſally, the Logarithm of CE to CQ is the Space BQEG.

785. From what was demonſtrated in (769.) is deduced an eaſy mechanical Method of *deſcribing an Ellipſis*, *viz.* fix a Pin in each Focus, and a Third in the End of one of the Diameters, and then tying a Thread faſt about the three Pins, take out that on the End of the Diameter, and placing the Head of it within the Thread, carry it with a tight and ſteady Hand about the two Foci, and it will deſcribe the Ellipſis very exactly. But the *organical Deſcription of Curves* is a Subject that will merit more particular Attention in another Part of this Work.

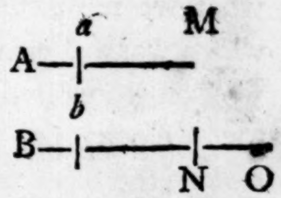
786. As to the Etymology of the Names of theſe *Conic Sections*, it is derived from the Nature of the Curves reſpectively; thus when the *Square of the Semi-ordinate is equal to the Rectangle under the Parameter and Abſciſſa*, *viz.* $yy = px$, then that Equality is intimate by the Name of the Curve, *viz.* Parabola. (740.) But, when that Rectangle (px) is leſs than the Square (yy) by the Quantity $\left(\frac{pxx}{a}\right)$ then that *Deficiency* is expreſſed in the Name of the Curve Ellipsis. (764.) By the Word Hyperbola, the *Exceſs* of the Rectangle px above the Square (yy) by the Quantity $\frac{pxx}{a}$ is ſhewn. *See* (765.) It is moreover obſervable, that $\frac{pxx}{a}$ is a Rectangle, *viz.* $\frac{px}{a} \times x$ ſimilar to the conſtant Rectangle $a \times p$; for $a : p :: x : \frac{xp}{a}$. We have now premiſed the *firſt Principles of Conics*, and can enlarge upon them at Pleaſure, or as we may at any Time hereafter have Occaſion.

THE

THE DOCTRINE of FLUXIONS, OR ELEMENTS of the NEW GEOMETRY.

THE Subject we are now entering upon is really a NEW GEOMETRY, not in the least known to the Ancients; *Algebra*, they are supposed to have understood; and kept it a Secret; but FLUXIONS remained among the *Incognita* of every Age from the Beginning of the World to our own. The Glory of this Invention was reserved for the great *British* Genius, Sir ISAAC NEWTON, as is put past all Doubt by the Author of a Treatise entitled, *Commercium Epistolicum*. As this Species of Mathesis opens at once an Entrance to all the sublimer Parts of Learning, it will be necessary here to premise the first Principles thereof in a Manner the most natural and easy that we can, and agreeable to that in which it was first proposed to the World by its Author. The Doctrine of Fluxions depends upon a few Principles only, and those easy to be understood, which here follow.

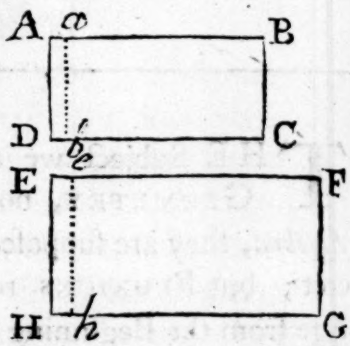
CHAP. XIII.



787. IF two Points A, B, be supposed to move equal Paces AM, BN, in the same Time, they are then said to *move with the same or equal Velocities*; and if in every equal Portion of Time, the Spaces passed over be equal, the *Velocity* is then said to be *equable* or *uniform*.

788. If

788. If in the ſame Time, the Point A paſſes from A to M, the Point B moves from B to O, then is the Velocity of the Point A to that of B, as A M to B O. Put $AM = x$, and $BO = y$. And let Aa, Bb, be the indefinitely ſmall Spaces deſcribed in the firſt Moment of Time; then the Motion or Velocities of the Points A and B being ſuppoſed equable, we ſhall have $Aa : AM :: Bb : BO$. Let $Aa = \dot{x}$, and $Bb = \dot{y}$; then ſhall $\dot{x}$, $\dot{y}$, (which are called the Fluxions of x and y) repreſent the Velocities with which the Points A, and B, every where move, and ſo it will be $x : y :: \dot{x} : \dot{y}$.

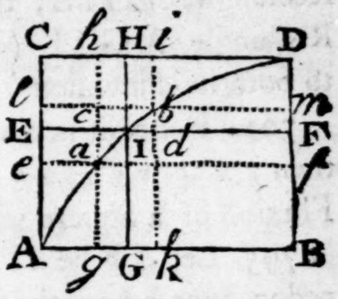


789. Since Superficies A B C D, E F G H, are deſcribed by the Parallel Motions of the Lines A D, E H, (615.) Let A a b D, and E e b H, be the indefinitely ſmall Parts deſcribed in the firſt Moment of Time in each. Put $AB = x$, $AD = a$; $EF = y$, and $EH = b$. Then alſo $Aa = \dot{x}$, and $Ee = \dot{y}$ (788.) The whole Rectangles are ax and by, and their *Naſcent Increments* (AabD, and EebH) are $a\dot{x}$ and $b\dot{y}$. If now the Times of deſcribing theſe Rectangles are equal, and the Velocities equable; then $a\dot{x} : b\dot{y} :: ax : by$; and ſo $a\dot{x}$, $b\dot{y}$, will be proportional to, and therefore may repreſent the Velocities with which thoſe Rectangles are generated, and are therefore their *Fluxions*. (788.)

790. But what we have now ſaid relates to the Fluxion or Flowing of the Rectangle *one Way only, or according to one Dimenſion*; but ſuppoſing it to increaſe according to both its Dimenſions, or ſo that both its Sides may flow to compleat the Space it contains, then we muſt add the Fluxions of both Sides together, and their Sum will be the Fluxion of the Rectangle in this Caſe.

791. In order to illuſtrate this it muſt be conſidered, that the illuſtrious Author of this Invention firſt delivered the Idea of what we now call a *Fluxion* under the Name of *Momentum*, which was a Term uſed in *Mechanics* to denote the Quantity of Motion generated by a given Quantity of Matter (A), and the Velocity *(a)* with which it moved conjointly. This *Momentum* therefore was properly repreſented by (A a); and if (Bb) denote any other *Momentum* reſulting from a Quantity of Matter

(B)



(B) moving with any Velocity as (*b*) then if theſe *Momenta* are generated in the ſame time, they will be as A *a* to B *b*; and if the Velocities of Motion are equal in both A and B, then will the *Momenta* be as the Quantities A and B. But inſtead of this mechanical Notation, we now uſe $x\dot{x}$ and $y\dot{y}$ for the *Momenta*, or Fluxions, generated by the Quantities x and y as above deſcribed.

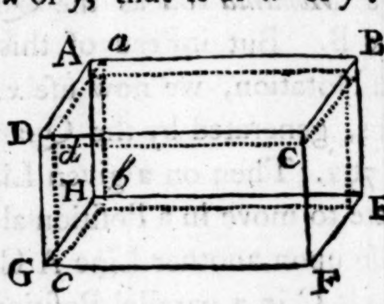
792. Then on a given Line A B, ſuppoſe a right Line G H were to move in a Poſition always parallel to itſelf, from A to B. Alſo upon another Line A C let the Line E F move from A towards C in a parallel Poſition alſo. And let the Velocities of the moving Lines be ſuch, that their Interſection at I ſhall deſcribe the Curve Line A I D. By the Motion of theſe Lines there will be deſcribed the Curvilineal Spaces A I D B and AIDC, alſo the Parallellograms AEFB and ACHG. Put AG = (E I =) x, and G I = (A E =) y; and the Fluxions of the Spaces deſcribed by the Generating Lines x and y, will be always as the Magnitude of thoſe Lines, ſuppoſing the Velocity of each to be conſtant. Therefore when the Line A C arrives to the Poſitions *g h*, G H, *i k*, the Fluxion of the curve-lined Space A I D B will be as *a g*, I G, and *b k*; but the Fluxion of the Parallellogram A E F B will be in each Poſition the ſame, being always as the equal Lines *c g*, I G, and *d k*. In the firſt Poſition therefore at *g h*, the Fluxion of the Space AIDB is leſs than that of the Parallellogram, becauſe *a g* is leſs than *c g*. In the third Poſition at *i k*, the Fluxion *b k* of the ſaid Space is greater than *d k* the Fluxion of the Parallelogram, conſequently there muſt be ſome intermediate Poſition G H where the Fluxion G I is the ſame in both.

793. In like manner it is ſhewn, that the Line which moves from A B to C D, when it is in the Poſitions *e f*, E F, and *l m* produces the Fluxions of the Space A I D C leſs, equal to, and greater than thoſe of the Parallellogram A G H C. But we have ſhewn that the Fluxion of the Rectangle A B F E is at the Poſition at I G equal to $y\dot{x}$, and of the Rectangle A G H C it is $x\dot{y}$; but theſe are alſo equal to the Fluxions of the two curvilineal Spaces A I G and A I E, which together make the

Rectangle A G I E; therefore the whole Fluxion of the said Rectangle A G I E (or $x\,y$) (considered as flowing according to both its dimensions) is equal $y\,\dot{x} + x\,\dot{y}$.

794. Hence it is manifest that when A G = A E, or $x = y$, then $y\,\dot{x} + x\,\dot{y} = 2\,x\,\dot{x}$, or $2\,y\,\dot{y}$; which therefore is as the Fluxion of a Square whose Side is x or y, that is of x^2 or y^2.

795. Let A F be a Parallelopipedon, whose Length is A B $= x$, its Breadth A D $= y$, and Depth A H $= z$. Suppose it to flow equably according to its Length only, by the Motion of the Plane A D G H $= y\,z$, and suppose



A a $(= \dot{x})$ the Space described in the Direction of A B the first Moment of Time; then will the Solid A c be as the Fluxion of the Solid A F, and therefore equal to $y\,z\,\dot{x}$. If the Solid were to flow only according to its Dimension of Breadth A B, by the Motion of the Plane A B E H $= x\,z$, then would the Fluxion be $x\,z\,\dot{y}$. And if the Solid flows in the Dimension of Depth A H, by the Plane A B C D $= x\,y$, then is its Fluxion $x\,y\,\dot{z}$. If then it flows in all its three Dimensions, the momentary Increase or Decrease will be proportional to the Sum of all, *viz.* $y\,z\,\dot{x} + x\,z\,\dot{y} + x\,y\,\dot{z}$, which therefore is the Fluxion of the whole Solid in this Case.

796. The same Thing may be otherwise proved thus: Put $x\,y = v$; then $x\,y\,z = v\,z$; and by (791, 792, 793,) $\dot{v} = \dot{x}\,y + x\,\dot{y}$, and the Fluxion of $v\,z$, *viz.* $\dot{v}\,z + v\,\dot{z}$ is = the Fluxion of $x\,y\,z$: Therefore, by Restitution, or writing $x\,y$ for v, and $\dot{x}\,y + x\,\dot{y}$ for $\dot{v}$, we shall have the Fluxion of $x\,y\,z = \dot{x}\,y\,x + x\,\dot{y}\,z + x\,y\,\dot{z}$ as before. And when $x = y = z$, then $x\,y\,z = x^3$ and its Fluxion $3\,x^2\,\dot{x}$, or $3\,x\,x\,\dot{x}$.

797. Since the Fluxion of x is $\dot{x}$ (788); and of x^2 is $2\,x\,\dot{x} = 2\,x^{2-1}\,\dot{x}$ (794); and of x^3, is $3\,x^2\,\dot{x}$ (796) $= 3\,x^{3-1}\,\dot{x}$; it is evident the Fluxion of x^m is $m\,x^{m-1}\,\dot{x}$; and the Fluxion of $x^m\,y^n$ is $m\,x^{m-1}\,\dot{x}\,y^n + n\,y^{n-1}\,\dot{y}\,x^m$.

798. Because $\sqrt{x} = x^{\frac{1}{2}}$ (for $x : \sqrt{x} :: \sqrt{x} : 1$, and $x^1 : x^{\frac{1}{2}} :: x^{\frac{1}{2}} : x^0$;) therefore the Fluxion of $\sqrt{x} = x^{\frac{1}{2}}$ is $\frac{1}{2}x^{\frac{1}{2}-1}\dot{x}$ $= \frac{1}{2}\,x^{\frac{1-2}{2}}\,\dot{x} = \frac{1}{2}\,x^{-\frac{1}{2}}\,\dot{x}$; and in like Manner the Fluxion

of

of $\sqrt[2]{x^3} = x^{\frac{3}{2}}$, is $\frac{3}{2} x^{\frac{1}{2}} \dot{x}$; and in general the Fluxion of $\sqrt[n]{x^m} = x^{\frac{m}{n}}$ is $\frac{m}{n} x^{\frac{m-n}{n}} \dot{x}$.

799. Thus also the Fluxion of $\sqrt[2]{xy} = x^{\frac{1}{2}} y^{\frac{1}{2}}$, is $\frac{1}{2} x^{-\frac{1}{2}} \dot{x} y^{\frac{1}{2}} + \frac{1}{2} y^{-\frac{1}{2}} \dot{y} x^{\frac{1}{2}}$, or $\frac{\dot{x} y^{\frac{1}{2}}}{2 x^{\frac{1}{2}}} + \frac{\dot{y} x^{\frac{1}{2}}}{2 y^{\frac{1}{2}}}$; and in general, the Fluxion of $\sqrt[m]{x^n y^p} = x^{\frac{n}{m}} y^{\frac{p}{m}}$, is $\frac{\overset{*}{p}}{m} x^{\frac{n-m}{m}} \dot{x} y^{\frac{p}{m}} + \frac{p}{m} y^{\frac{p-m}{m}} \dot{y} x^{\frac{n}{m}}$. $*\frac{n}{m}$

800. The Fluxion of any Root of *Surd Compound* Quantities are also thus easily obtained. As the Fluxion of $\sqrt[2]{\overline{xy+yy}} = \overline{xy+yy}^{\frac{1}{2}}$ is $\frac{1}{2} \overline{xy+yy}^{-\frac{1}{2}} \times \overline{x\dot{y} + y\dot{x} + 2y\dot{y}} = \frac{x\dot{y} + y\dot{x} + 2y\dot{y}}{2\,\overline{xy+yy}^{\frac{1}{2}}}$. And universally the Fluxion of $\sqrt[q]{a^n + x^m} = \overline{a^n + x^m}^{\frac{1}{q}}$ is $\frac{1}{q} \overline{a^n + x^m}^{\frac{1-q}{q}} \times m x^{m-1} \dot{x}$.

Also $x^p \sqrt[q]{y^n + z^m} = x^p \times \overline{y^n + z^m}|^{\frac{1}{q}}$ gives the Fluxion $p x^{p-1} \dot{x} \times \overline{y^n + z^m}|^{\frac{1}{q}} + x^p \times \frac{\overline{n y^{n-1} \dot{y} + m z^{m-1} \dot{z}}}{q \times \overline{y^n + z^m}|^{\frac{q-1}{q}}}$

801. The Fluxion of a Fraction $\frac{x}{y}$ is thus found. Put $\frac{x}{y} = v$, then $x = vy$, and the Equality between x and vy ought ever to hold in all their State of Increase or Decrease, their Fluxions will always be equal, but the Fluxion of x is $\dot{x}$, and the Fluxion of vy is $v\dot{y} + y\dot{v}$ (792, 793); therefore $\dot{x} = v\dot{y} + y\dot{v}$; and so $\dot{x} - v\dot{y} = y\dot{v}$, and $\dot{v} = \frac{\dot{x} - v\dot{y}}{y} = \frac{y\dot{x} - x\dot{y}}{yy}$ by substituting for v its Value $\frac{x}{y}$.

802. Because (a) represents a *constant Quantity*, its Fluxion is (o), therefore the Fluxion of $a + x$ is only $\dot{x}$, and the same may be said of numerical Quantities, thus the Fluxion of $1 + x$ is only $\dot{x}$. The Fluxion of $a\,x$, or $2\,x$, is only $a\,\dot{x}$ or $2\,\dot{x}$. Hence the Fluxion of $\frac{a}{x}$ is $\frac{-a\,\dot{x}}{x\,x}$, and that of $\frac{x}{a + x}$ is $\frac{a\,\dot{x}}{aa + 2\,a\,x + x\,x}$. But the Fluxions of complicated Fractions are best found in their Value expressed in infinite Series. Thus $\frac{a\,a}{b + x} = \frac{a^2}{b} - \frac{a^2\,x}{b^2} + \frac{a^2\,x^2}{b^3}$, &c. (103); the Fluxion of which Series is $-\frac{a^2\,\dot{x}}{b^2} + \frac{2\,a^2\,x\,\dot{x}}{b^3} - \frac{3\,a^2\,x^2\,\dot{x}}{b^4}$, &c. (by 801).

803. But the principal Use of Fluxions is to find the *Fluent* or *flowing Quantity* from the *Fluxion given*, which is done in general by the Reverse of the Method of finding the Fluxion of a given Fluent, above delivered. This Method consists in three Particulars, *viz.* (1) Expunge the fluxionary Letter; thus the Fluxion $2\,x\,\dot{x}$ becomes $2\,x$. (2) Add Unity to the Index or Exponent of the flowing Quantity; and thus $2\,x$ becomes $2\,x^{1+1} = 2\,x^2$. (3) Divide by the Exponent thus encreased, and so $2\,x^2$ becomes x^2, which is the Fluent of the Fluxion $2\,x\,\dot{x}$, as required. Thus $3\,x^2\,\dot{x}$ is (1) $3\,x^2$, (2) $3\,x^3$, and (3) x^3. So in general $m\,x^{m-1}\,\dot{x}$ becomes first $m\,x^{m-1}$, then $m\,x^m$, and lastly x^m the Fluent required.

804. Hence also the Fluent of $x\,\dot{x}$ is $\frac{x\,x}{2}$; for it is first x, then x^2; and lastly $\frac{x^2}{2}$. So $x^2\,\dot{x}$ is first, x^2; secondly, x^3; thirdly $\frac{x^3}{3}$ the flowing Quantity; and by Inspection it appears that the Fluent of $\dot{x}$ is x; of $x\,\dot{y} + y\,\dot{x}$ is $x\,y$ (by 792, 793,) and the Fluent of $x\,y\,\dot{z} + x\,z\,\dot{y} + \dot{x}\,y\,z$ is $x\,z\,y$ (by 796.) And also that the Fluent of $\frac{x\dot{y} - y\dot{x}}{y\,y}$ is $\frac{x}{y}$; and that the Fluent of $\dot{x} + x\,\dot{x} + x^2\,\dot{x}$, &c. is $x + x^2 + x^3$, and in general

neral the Fluent of $m x^{m-1} \dot{x}$ will be x^m; and of $-mx^{-m-1}\dot{x}$ will be x^{-m}, or $\frac{1}{x^m}$ (798).

805. Thus also the Fluent of $\frac{m}{n} x \dot{x}^{\overline{\frac{m-n}{n}}}$ is $x^{\frac{m}{n}} = \sqrt[n]{x^m}$, and of $a x^{\frac{m}{n}} \dot{x}$ is $\frac{an}{m+n} x^{\overline{\frac{m+n}{n}}}$ for by rejecting $\dot{x}$, it is $a x^{\frac{m}{n}}$, and by adding 1 to the Index $\frac{m}{n}$ it is $a x^{\overline{\frac{m+n}{n}}}$ (for $\frac{m}{n} + 1 = \frac{m+n}{n}$;) and dividing by $\frac{m+n}{n}$, we have $\frac{an}{m+n} x^{\overline{\frac{m+n}{n}}}$ as above. Now this is a general Form for any other fluxionary Power or Root, as $4\sqrt{x\dot{x}} = 4x^{\frac{1}{2}}\dot{x}$, where $a = 4$, $m = 1$, $n = 2$, and so $\frac{m+n}{n} = \frac{3}{2}$, therefore $\frac{an}{m+n} = \frac{8}{3}$; and consequently we shall have $\frac{an}{m+n} x^{\overline{\frac{m+n}{n}}} = \frac{8}{3} x^{\frac{3}{2}} = \frac{8}{3} \sqrt[2]{x^3}$ for the Fluent of $4x^{\frac{1}{2}}\dot{x}$. After this Manner the Fluent of $\sqrt[3]{x^5 \dot{x}} = x^{\frac{5}{3}}\dot{x}$ will be found $\frac{3}{8} x^{\frac{8}{3}} = \frac{3}{8}\sqrt[3]{x^8}$; and of $x^{\frac{1}{2}}\dot{x} = 1x^{-2}\dot{x}$, will be $-x^{-1}$; of $\frac{1}{\sqrt{x^3}}\dot{x} = x^{-\frac{3}{2}}\dot{x}$, will be $-\frac{2}{1}x^{-\frac{1}{2}} = \frac{2}{-\sqrt{x}}$.

And lastly, the Fluent of $\frac{1}{x}\dot{x} = x^{-1}\dot{x}$ will be $\frac{1}{0}x^{\frac{0}{1}} = \frac{1}{0}x^0 = \frac{1}{0} \times 1 = \frac{1}{0} =$ Infinity; since here $a = 1$, $m = -1$, and $n = 1$.

806. If a Fluxion be multiplied into some Quantity affected with a *Vinculum*, that is, into some Compound or Surd Quantity, which Quantity is the Fluent of the said Fluxion, as $a\dot{x} \times \overline{aa + ax}|^m$, then will the Fluent of the Expression be had by the general Rule thus; strike out the Fluxion, and it becomes $\overline{aa + ax}|^m$; and add Unity to the Index m, and it is $\overline{aa + ax}|^{m+1}$; then divide by the Index so increased, and you have

have the Fluent $\frac{1}{m+1} \times \overline{aa+ax}|^{m+1}$ for the Expression proposed.

807. Thus also $a\dot{x} \times \sqrt[n]{aa+ax}|^m = a\dot{x} \times \overline{aa+ax}|^{\frac{m}{n}}$, by the very same Proceedure, will have its Fluent thus expressed $\frac{n}{n+m} \overline{aa+ax}|^{\frac{m+n}{n}}$. Thus the Fluent of $2x\dot{x} \times \sqrt{aa+xx} = 2x\dot{x} \times \overline{aa+xx}|^{\frac{1}{2}}$ is $\frac{2}{3}\overline{aa+xx}|^{\frac{3}{2}}$, or in Surds $\frac{2aa+2xx}{3}\sqrt{aa+x^2}$. So the Fluent of this Expression $mx^{m-1}\dot{x} \times \overline{a^q+x^m}|^n$ is $\frac{n+1}{1}\overline{a^q+x^m}|^{n+1}$; and so for any other of this kind.

808. If the Fluxion without the *Vinculum* be not that of the Quantity under it, but is yet *in some given Ratio* to it, the Fluent may in this Case be had in a finite Number of Terms. Suppose the Expression were $x\dot{x}\sqrt{aa+x^2}$, or $x\dot{x} \times \overline{aa+xx}|^{\frac{1}{2}}$ where the Fluxion $x\dot{x}$ is to that of x^2, as 1 to 2, *viz.* $x\dot{x} : 2x\dot{x} :: 1 : 2$. It will be best here to proceed by Substitution; therefore let $\sqrt{aa+x^2} = z$; then $aa+xx = zz$, and $2x\dot{x} = 2z\dot{z}$, or $x\dot{x} = z\dot{z}$; therefore $x\dot{x}\sqrt{aa+xx} = zz\dot{z}$; but the Fluent of $zz\dot{z}$ is $\frac{1}{3}z^3 = \frac{1}{3}zz \times z =$ (by Substitution) $\frac{aa+xx}{3}\sqrt{aa+xx}$.

809. Also the Fluent of $x^{m-1}\dot{x} \times \overline{aa^q+x^m}|^n$ may be found in like Manner. For here the Fluxion $x^{m-1}\dot{x}$ is to that under the *Vinculum* of x^m, *viz.* $mx^{m-1}\dot{x}$, as 1 to m; which Ratio is given. Therefore put $\overline{aa^q-x^m}|^n = z^n$, and so $aa^q+x^m = z$, and consequently $mx^{m-1}\dot{x} = \dot{z}$, and therefore $x^{m-1}\dot{x} = \frac{\dot{z}}{m}$, therefore $x^{m-1}\dot{x} \times \overline{aa^q+x^m}|^n = \frac{z^n\dot{z}}{m}$. But the Fluent of the Latter, *viz.* $\frac{z^n\dot{z}}{m} = \frac{1}{m} \times z^n\dot{z}$ is $\frac{1}{mn+n} z^{n+1} =$

$\frac{1}{mn+n}\overline{\times aa^q x^m}|^{n+1}$ = the Fluent of the given Expression.

810. Thus also the Fluent of $dx^{m-1}\dot{x}\times\overline{a+fx^m}|^n$ may be found; for put $\overline{a+fx^m}|^n = z^n$, then $\overline{a+fx^m} = z$, and so $\dot{z} = fmx^{m-1}\dot{x}$, and therefore $\frac{\dot{z}}{fm} = x^{m-1}\dot{x}$, and so $\frac{d\dot{z}}{fm} = dx^{m-1}\dot{x}$; therefore $dx^{m-1}\dot{x}\times\overline{a+fx^n}|^n = \frac{d\dot{z}}{fm}\times z^n = \frac{d}{fm}\times z^n\dot{z}$, but the Fluent of this latter Fluxion is $\frac{d}{fm}\times\frac{1}{n+1}\times z^{n+1}$, or $\frac{d}{fm\times\overline{n+1}}z^{n+1}$; if therefore instead of z we substitute its Equal $\overline{a+fx^m}$, we have $\frac{d}{fm\times\overline{n+1}}\times\overline{a+fx^m}|^{n+1}$ for the Fluent of the Expression proposed.

811. The last is a *general Form* for any fluxional Expression of that Kind; thus suppose the Fluent of $\dot{x}\times\overline{a+x}$ were required; here $m=1$, $n=1$, $f=1$, $dx^{m-1} = dx^1 = d = 1$, therefore $\frac{d}{fm\times\overline{n+1}} = \frac{1}{2}$, and consequently $\frac{1}{2}\overline{a+x}|^2$, will be the Fluent required. Again, let $x\dot{x}\sqrt{aa+x^2} = x\dot{x}\times\overline{aa-x^2}|^{\frac{1}{2}}$; here $m=2$, $n=\frac{1}{2}$, $f=1$, and $dx^{m-1} = x = dx$, and therefore $d=1$; hence $\frac{d}{fm\times\overline{n+1}} = \frac{1}{2\times\frac{3}{2}} = \frac{1}{3}$, and so $\frac{1}{3}\overline{aa+x^2}|^{\frac{3}{2}}$ or $\frac{aa+x^2}{3}\sqrt{aa+x^2}$ is the Fluent, the same as before.

812. The Fluents of such fluxional Expressions may be found also by an infinite Series; for $\overline{a+fx^m}|^n$, thrown into an infinite Series will be $a^n + \frac{n}{1}fx^m a^{n-1} + \frac{n}{1}\times\frac{n-1}{2}\times ffx^{2m}a^{n-2}\times$, &c. Or if $A = \frac{1}{a}$, $B = A\frac{n}{a}$, $C = B\frac{n-1}{2a}$, $D = C\frac{n-2}{3a}$, &c. we shall have $\overline{a+fx^m}|^n = a^n\times 1 + n$

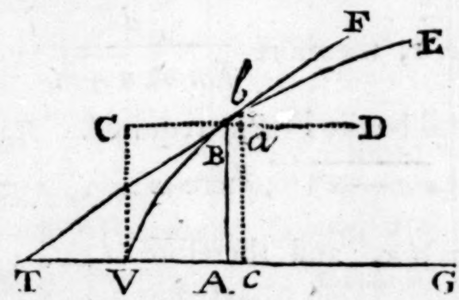
Afx^m

$A f x^m + \frac{n-1}{2} B f f x^{2m} + \frac{n-2}{3} C f^3 x^{3m} +$, &c. Now if this Series be multiplied by the Fluxion $d x^{r-1} \dot{x}$, and the Fluent of each Term be taken, we shall have $a^n d x^r \times \frac{1}{r}$

$$+ \frac{n}{r+m} A f x^m + \frac{n-1}{r+2m} \frac{1}{2} B f^2 x^{2m} + \frac{n-2}{r+3^m} \frac{1}{3} C f^3$$

x^{3m}, &c. for the Fluent of $d x^{r-1} \dot{x} \times \overline{a + f x^m}|^n$ which therefore will serve as a general Form for all binomial Expressions of Fluxions of the same Kind with this. But where the Fluent can be had in finite Terms, the general Form of the (810) is much the best.

813. But it must be confessed there is no direct and certain Method of finding the Fluent of a given Fluxion that shall hold in every Case, because there may be different Fluents which produce, and consequently which belong to the same Fluxion; thus x, $a + x$, $x + 1$, &c. all have the common Fluxion $\dot{x}$; and if there be nothing in the Circumstances or Conditions of the Problem to determine the Fluent, we are wholly uncertain what it is. But of this more hereafter.



814. To find the Fluxion of *Curve Lines* and *Curvilineal Spaces*, or Areas, there is required but one *Postulatum*, viz. *that the Curve* V B E *and its Tangent* T F *at the Point of Contact* B, *are in the same Direction for an indefinite small Distance on either Side, as* B *b*. *Or, that the small Portion of the Curve and Tangent* B *b are coincident, and make a Right Lineola*; then putting the Abscis V A $= x$, the Ordinate A B $= y$, and the Curve V B $= z$, and draw $c b$ infinitely near and parallel to A B, and C B a parallel to the Axis T G. If now we suppose the Ordinate A B to move equably along the Axis, and with its Point B to describe the Curve V B E, then it is evident in any Situation of the Ordinate A B, the Fluxion, or Velocity, of the Abscis A V, is A $c = \dot{x}$; and of the Ordinate A B, it is $a b = \dot{y}$; and of the Curve V B, it is B $b = \dot{z}$.

815. Now

815. Now because of parallel Lines, the Triangles B $a b$ and T A B are similar; and therefore (657) $TA : AB :: Ba : ab :: \dot{x} : \dot{y}$. Let the Curve be the *Parabola*; then $TA = 2x$, (745) and so $2x : y :: \dot{x} : \dot{y}$ or $2x\dot{y} = y\dot{x}$; but $px = yy$, (740) and putting $p = 1$, it is $x = yy$, and so $2x\dot{y} = 2yy\dot{y} = y\dot{x}$, or $2y\dot{y} = \dot{x}$. Whence it appears again, that as the Fluxion of x is $\dot{x}$, so the Fluxion of yy is $2y\dot{y}$ agreeably to (794).

816. Because the Ordinate A B is at Right Angles with the Axis, the small Triangle B $a b$ is rectangular, and so $Ba^2 + ab^2 = Bb^2$, $\dot{x}^2 + \dot{y}^2 = \dot{z}\dot{z}$, consequently the Fluxion of any Curve V B is $\dot{z} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$.

817. Again it is evident, the Fluxion of the curvilineal Space A V B is the same with that of the Parallellogram A V C B, at the Instant the Ordinate arrives at the Situation A B; for at any Time before it was less, and at any Time after, it is greater. But the Fluxion of the Parallellogram is as $A\,c\,a\,B = y\dot{x}$, (792) which therefore is the Expression for the Fluxion of any curvilineal Space.

CHAP. XIV.

Of the METHOD *de* MAXIMIS *and* MINIMIS; *the* RECTIFICATION *of* CURVES; *the* QUADRATURE *of* Curvilineal Spaces; *and the* CUBATURE *of* SOLIDS.

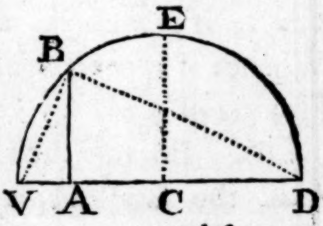
818. WE shall now give a Specimen of the universal Application of Fluxions to all mathematical Purposes; and which will contain the original Principles of the most extensive Geometry: We shall illustrate this Subject in the following capital Articles.

I. *The* METHOD *de* MAXIMIS *and* MINIMIS.

This Method consists in determining an *extreme Value* of any proposed Quantity, that is, to find when it is *greatest*, or *least*.

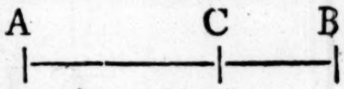
Now, as such a Quantity becomes greater or lesser *by flowing*, so when it has attained its greatest or least Magnitude, that flowing State is at an End, and its Fluxion in that Case must of Course be nothing, or $= 0$.

819. To illustrate this, let VBD be a Circle; $VD = a =$ Diameter, the Abscis $VA = x$, the Ordinate $AB = y$; then $AD = a - x$, and because of the Rect-angle VBD, (645) it is $AD : AB : AB : AV$, that is $a - x : y :: y : x$, and so $ax - xx = yy$. Now to find when AB or y is greatest, put the Equation into Fluxions, and it is $a\dot{x} - 2x\dot{x} = 2y\dot{y}$. And because in this Case $\dot{y} = 0$, (818) therefore $2y\dot{y} = 0$, and so $a\dot{x} - 2x\dot{x} = 0$. Hence $a\dot{x} = 2x\dot{x}$, that is, $a = 2x$, or $x = \frac{1}{2}a = VC$, when y or AB is a *Maximum*, as it evidently is at CE.

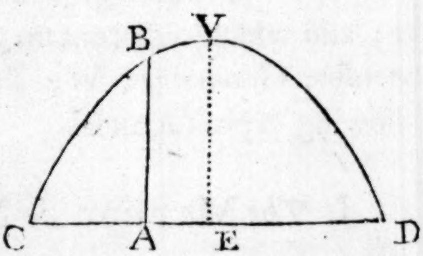


820. Hence it appears, that when a Line VD is divided into two Parts, AV and AD, whose Product $AV \times AD$ is a *Maximum*, those Parts are *equal to each other*, or the given Line is *bisected*.

821. Let the right Line AB be so divided in C, that $AC^m \times CB^n$, may be a *Maximum*; put $AC = x$, and $CB = y$; then it is evident, that the Fluxion of both Parts is the same, and that while AC increases, BC decreases; therefore $\dot{x} = -\dot{y}$. Since $x^m y^n$ is a *Maximum*, its Fluxion $mx^{m-1}\dot{x} \times y^n - ny^{n-1}\dot{y} \times x^m = 0$; whence $mx^{m-1}y^n = ny^{n-1}x^m$; therefore $x^m : y^n :: mx^{m-1} : ny^{n-1} :: \frac{m}{x} \times x^m : \frac{n}{y} \times y^n$; therefore $\frac{m}{x} = \frac{n}{y}$, and $nx = my$, whence $x : y :: m : n$; or the Parts are directly as their Powers.



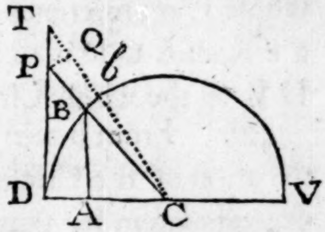
822. Again, let CVD be a Parabola; and put $CD = a$, $CA = x$, $AB = y$, then $AD = a - x$, and (762) $p \times AB = CA \times AD$, or $py = ax - xx$; and in Fluxions $p\dot{y} = a\dot{x} - 2x\dot{x}$. When therefore AB or y is a *Maximum*, it is $\dot{y} = 0$; then $a\dot{x} = 2x\dot{x}$, or $x = \frac{1}{2}a = CE$, as before.



II. *The*

II. *The* RECTIFICATION *of* CURVES.

823. By the *Rectification* of a Curve, is meant no more than the finding a *right Line equal to it*. I shall give an Example in the Arch D B of the Circle VBD. Let T D be a Tangent at D, draw C P, and infinitely near it C T; and with the Radius C P describe the small Arch P Q. Put D C $= a$, D P $= x$, and then T P $= \dot{x}$, and B $b = \dot{z}$ = Fluxion of the Arch D B. Then it is TP : QP : : CP : CD; and P Q : B b : : CP : CB = CD. Therefore (*ex Æquo* 652,) TP : B b : : CP^2 : CD^2; but $CP^2 = a^2 + x^2$; therefore $a^2 + x^2 : a^2 :: \dot{x} : \dot{z}$; whence $\dot{z} = \frac{a^2 \dot{x}}{a^2 + x^2}$;



now $\frac{a^2}{a^2 + x^2} = 1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6}$, &c. which multiplied by $\dot{x}$, makes $\dot{x} - \frac{x^2 \dot{x}}{a^2} + \frac{x^4 \dot{x}}{a^4} - \frac{x^6 \dot{x}}{a^6}$, &c. $= \frac{a a \dot{x}}{a a + x^2} = \dot{z}$, the Fluent of which is $x - \frac{x^3}{3 a^2} + \frac{x^5}{5 a^4} - \frac{x^7}{7 a^6}$, &c. = DB the Arch required in the Values of the right Lines C D, and D P.

824. By putting CD $= a = 1$, the Length of the Arch D B will be thus expressed, $x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7$, &c. And if the Arch B D = 30 Degrees, then A B $= \frac{1}{2}$ C D $= \frac{1}{2}$, * and A C^2 = C B^2 − A B^2 $= 1 - \frac{1}{4} = \frac{3}{4}$, whence A C $= \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$. And since C A : A B : : C D : D P, or $\frac{1}{2}\sqrt{3} : \frac{1}{2} :: 1 : x = \frac{1}{\sqrt{3}}$, and so $x^2 = \frac{1}{3}$, and $x^3 = \frac{1}{3}\sqrt{\frac{1}{3}}$, $x^5 = \frac{1}{9}\sqrt{\frac{1}{3}}$, $x^7 = \frac{1}{27}\sqrt{\frac{1}{3}}$, &c. Therefore by substituting these Values of x, we have the Arch D B $= \sqrt{\frac{1}{3}} - \frac{1}{9}\sqrt{\frac{1}{3}} + \frac{1}{45}\sqrt{\frac{1}{3}} - \frac{1}{189}\sqrt{\frac{1}{3}}$, or D B $= \sqrt{\frac{1}{3}} \times \overline{1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729}}$, &c.

825. Now since D B is $\frac{1}{6}$ of the Semi-circle, therefore $6\sqrt{\frac{1}{3}} = \sqrt{\frac{1}{12}} = 2\sqrt{3} = 3{,}464101615$; and hence, if each Term

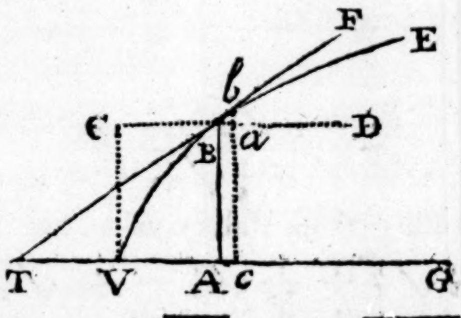
* For the Side of an Hexagon is the Chord of 60 Degrees, and equal to the Radius; Half that Chord is the Sine of 30 Degrees, which is therefore equal to Half the Radius.

Term in the above Series be multiplied by this Number, and the Sum of all the negative Products taken from the Sum of all the affirmative Ones, the Residue will give the Value of the whole Circumference = 3,14159265358979, *&c.* Wherefore the Radius C D is to the Semi-circle D B V (or the Diameter D V to the whole Circle) as 1 to 3,14159, *&c.*

826. From hence appears the Impossibility of an exact Rectification of the Circle; and since the Quadrature of the Circle depends upon it, that also is impossible; in vain therefore must any one attempt to *rectify, or square the Circle*, tho' we can approximate to the Truth, as there can ever be Occasion for. The following prodigious Number, calculated by the late Mr. *Sharp*, viz. 3,1415 9265 3589 7932 3846 2643 3832 7950 2884 1971 6939 9375 1058 2097 4944 5923 0781 6405; and afterwards examined to the last Figure by Mr. *Machin*, is more than sufficient for computing the Number of Grains of Sand that may be contained within the Sphere of the fixed Stars. I give this Number a Place here as it is the Product of the greatest Effort that was ever made in practical Geometry. The Rectification of the *Conic Sections*, and other Curves, will hereafter find a Place, when we have Occasion to apply them to Use.

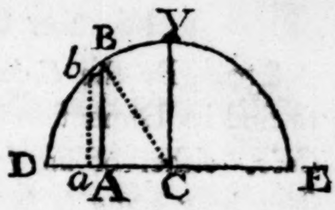
III. *The* QUADRATURE *of* CURVILINEAL SPACES.

827. To square any Curve-lined Space, is to find an Expression of a Right-lined Space equal to it. Thus, suppose it is required to square the *Parabolic Area* ABV. Then because $px = yy$, (658) we have $y = \sqrt{px} = p^{\frac{1}{2}} x^{\frac{1}{2}}$, which multiplied by $\dot{x} = Ac$, gives $y\dot{x} = p^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} = AcbB$, the Fluxion of the Space. (789) The

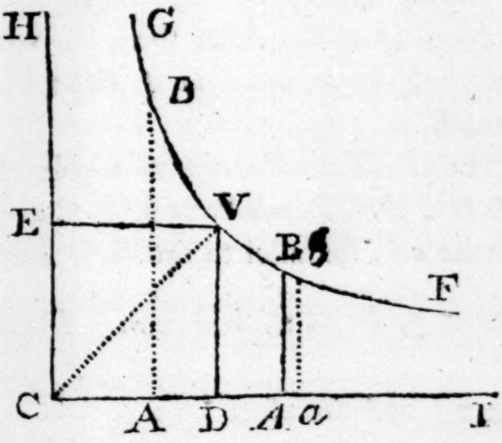


Fluent of which is $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$ (803) $= \frac{2}{3}\sqrt{px^3} = \frac{2}{3}\sqrt{px \times x^2}$ $= \frac{2}{3} = \sqrt{y^2 \times x^2} = \frac{2}{3} yx$. But CB = VA = x, CV = AB = y; *wherefore the Area* A B V *of the Parabola is* $\frac{2}{3}$ *of its circumscribing Parallellogram* A B C V. And consequently the external Space VBC is $\frac{1}{3}$ thereof.

828.



828. To square the Circle: Let the Ordinate A B move from V C equably toward D put C V $=$ B C $= a$, A C $= x$, A B $= y$, then $AC^2 + AB^2 = BC^2$, or $aa = x^2 + y^2$. Therefore $aa - x^2 = y^2$, and $\sqrt{a^2 - x^2} = y$, and $\dot{x}\sqrt{aa - x^2} = y\dot{x} = AabB$, the Fluxion of the Area ABVC, (789). But $\sqrt{aa - xx} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$, &c. therefore $\dot{x}\sqrt{aa - x^2} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6 x}{16a^5}$, &c. $= y\dot{x}$; the Fluent of which Series is $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5}$, &c. $=$ A B V C, the Space required in Terms of the right Line A C. When A B arrives at D and vanishes, then $a = x =$ C D $= 1$, and the Series will become $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} -$, &c. $=$ the Quadrant V C D; or if the Diameter $= 1 =$ D E, then this Series will be the Area of the whole Circle, and will be $= 0{,}785398$, &c.



829. To square the *Hyperbolic Space* DVBA; let C D $= a$, D A $= x$, A B $= y$; then it is $y = \frac{aa}{a + x}$, (or putting $a = 1$,) $y = \frac{1}{1 + x}$. (239) The Fluxion of which is $\dot{x}\,y = \frac{\dot{x}}{1 + x} =$ A a b B the Fluxion of the Space A D V B. But $\frac{1}{1 + x} = 1 - x + x^2 - x^3$, &c. (as will appear by Division) therefore $\frac{\dot{x}}{1 + x} = \dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x}$, &c. the Fluent of which is $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. $=$ A D V B, the Space required.

IV.

IV. *The* CUBATURE *of* SOLIDS.

830. By the *Cubature of a Solid* is meant no more than to find its Dimension or Content estimated in the Measure of a *Cube*, either of an *Inch*, *Foot*, *Yard*, *Mile*, *&c.* that is, to shew how many such Cubes are contained in, or are equal to the Bulk of the given Solid; to which Purpose the following Theorems are premised.

In the Circle D B V (Fig. to Art. 823.) the infinitely small Sector C B b may be esteemed a *rectilineal Triangle*, and so equal to $\frac{1}{2}$ B C $\times$ b B, or $\frac{1}{2}\, a\, \dot{z}$, the Fluent of which is $\frac{1}{2}\, a z =$ the Sector C D B; and putting $p =$ Periphery of the Circle, when $z = p$, then the Area of the Circle is $\frac{1}{2}\, a p$. This being premised, we proceed

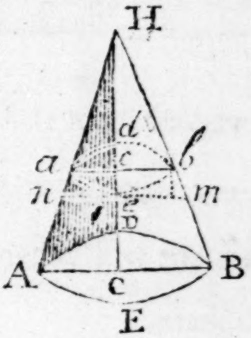
831. To *cube the* CYLINDER A G. Put A H $= a$, and the Circumference $= p$; then will the circular Area be $\frac{1}{2}\, a p$, and let A E $= x$; then $\frac{1}{2}\, a p \dot{x}$ will be the Fluxion of the Cylinder, and the Fluent thereof $\frac{1}{2}\, a p x =$ the Solidity of the Cylinder.

832. Also $p \dot{x} =$ Fluxion of the cylindric Surface, and its Fluent ($p x$) the Area thereof; whence $\frac{1}{2} a \times p x =$ the *Solidity of the Cylinder, equal to its Superficies multiplied by half its Diameter.*

833. To *cube the triangular* PYRAMID A B C D. Put A B $= a$, B C $= b$, B D $= d$; make $a\, b\, c$ parallel to the Base A B C; and put $a b = y$, $b c = z$, b D $= x$. Then $a : b :: y : \frac{b y}{a} = z$; and therefore $\frac{1}{2}\, y z = \frac{b y y}{2 a} =$ the Area of the flowing Triangle $a\, b\, c$; and $\frac{b y y \dot{x}}{2 a} =$ the Fluxion of the Pyramid; again $c : a :: x : \frac{a x}{c} = y$, and $y y = \frac{a^2 x^2}{c^2}$, therefore the Fluxion of the Pyramid is $\frac{b a^2 x^2 \dot{x}}{2 a c^2} = \frac{a b x^2 \dot{x}}{2 c^2}$, whose Fluent is $\frac{a b x^3}{6 c^2}$ = solid Content of the Pyramid $a\, b\, c$ D. And when x becomes

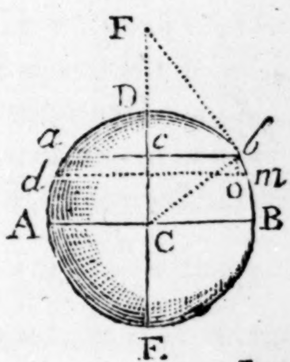
c,

c, or $bD = BD$, then $\frac{abx^3}{6c^3} = \frac{abc}{6} =$ the Solidity of the Pyramid ABCD. Hence every Prism is equal to its Base $\frac{1}{2}ab$ multiplied by $\frac{1}{3}c =$ its Altitude.



834. To *cube the right* CONE AHB. Let AC $= a$, $CH = b$, and $p =$ Periphery of the Base; suppose the Section $adbe$ parallel to the Base ADBE; and make $ac = y$, $cH = x$; then will $a : p :: y : \frac{py}{a} =$ Periphery of the Circle $adbe$. And its Area will be $\frac{py}{a} \times \frac{1}{2}y$ (830) $= \frac{\frac{1}{2}pyy}{a}$; whence $\frac{\frac{1}{2}pyy\dot{x}}{a}$ will be the Fluxion of the Cone; and since $b : a :: x : \frac{ax}{b} = y$, we have $yy = \frac{a^2x^2}{bb}$; and the Fluxion above will become $\frac{\frac{1}{2}pa^2x^2\dot{x}}{ab^2} = \frac{\frac{1}{2}pax^2\dot{x}}{b^2}$; the Fluent of which is $\frac{pax^3}{6b^2}$; and when $x = b$, then the Fluent is $\frac{pab}{6} =$ the Solidity of the Cone AHB.

835. Because $\frac{pab}{6} = \frac{1}{3}$ of $\frac{pab}{2}$, it appears that the Cone is *one Third of a Cylinder, having the same Base and Altitude.* (831) Also because $\frac{pab}{6} = \frac{pa}{2} \times \frac{b}{3}$, it appears, *the Solidity of a Cone is equal to the Base multiplied into one third Part of its Height.*



836. *To cube the Sphere* ADBE. Let AC $= a$, and $p =$ Periphery of the Circle, whose Diameter is AB. Draw $ab \parallel AB$, and put $ac = y$; then $a : p :: y = \frac{py}{a}$, the Periphery of a Circle on the Diameter ab; therefore $\frac{pyy}{2a} =$ Area of the said Circle.

Let

Let $Dc = x$; then $\frac{pyy\dot{x}}{2a}$ is the Fluxion of the Segment aDb. But $yy = 2ax - xx$ (819), therefore $\frac{p\dot{x}}{2a} \times yy = \frac{p\dot{x}}{2a} \times 2ax - xx = \frac{2pax\dot{x} - pxx\dot{x}}{2a} =$ Fluxion of the Segment aDb; whose Fluent is $\frac{2pax^2}{4a} - \frac{px^3}{6a} =$ Solidity of the said Segment. And when $x = 2a = d = DE$, the Diameter, then the said Fluent will become $\frac{pd^3}{2d} - \frac{pd^3}{3d} = \frac{pd^2}{6} =$ the Solidity of the Sphere.

837. Now since $\frac{pd^2}{6} = \frac{4a^2p}{6} = \frac{2}{3}a^2p = \frac{1}{3}a \times 2ap = \frac{1}{3}a \times dp$ it is evident, *the Solidity of a Sphere is equal to the Rectangle under the Diameter and Circumference of its greatest Circle multiplied into* $\frac{1}{3}$ *of its Semidiameter.* Also since $\frac{pd^2}{6} = \frac{2}{3} \times \frac{pd^2}{4}$, and $\frac{pd}{4} \times d = \frac{1}{2}pa \times d$; *therefore every Sphere is equal to* $\frac{2}{3}$ *of its circumscribing Cylinder* (831).

838. To *square the Superficies of the Sphere*; draw the Tangent Fb meeting the Axis produced in F, and join Cb; draw $bo \parallel DE$, and dm infinitely near, and parallel to ab; then are the Triangles Cbc and CFb similar (659.) And so cb; $Cb :: bo : bm$, or $y : a :: \dot{x} = \frac{a\dot{x}}{y} = \dot{z} = bm$, the Fluxion of the Arch Db; which multiplied by $\frac{py}{a}$ (836,) gives $\frac{apy\dot{x}}{ay} = p\dot{x} =$ the Fluxion of the Superficies of the Segment aDb, whose Fluent $px =$ Area thereof, and when $x = d$, the Superficies of the whole Sphere will be pd, as required.

839. Hence the Superficies of a Segment is to that of the Sphere, as px to pd, or as x to d, that is, as the Altitude Dc of the Segment to the Diameter DE of the Sphere. And since $\frac{1}{2}pa = \frac{pd}{4}$ (830) $= \frac{1}{4} \times pd$; it appears *the Superficies of the Sphere is equal to four times the Area of its greatest Circle*. Hence, when $d = 1$, we have $p = 3,14159265$ (825).

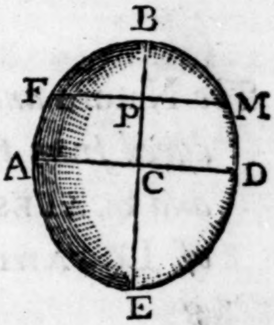
840.

840. If we put D, P, for the Diameter, and Periphery of one Circle, and d, p, for the same in another; and since $D : d :: P : p$; and the Areas being denoted by A and a, we have $A : a :: \frac{1}{4} DP : \frac{1}{4} dp :: DP : dp :: DD : dd$; *that is, the Areas of Circles are as the Squares of their Diameter.*

841. Let S and s denote the Solidity of two Spheres. Then $S : s :: \frac{1}{6} PD^2 : \frac{1}{6} pd^2 :: PD^2 : pd^2 :: D^3 : d^3$; (836) that is, *the Spheres are to each other as the Cubes of their Diameter.*

842. Let the Superficies of two Spheres be denoted by F and f. Then $F : f :: PD : pd :: D^2 : d^2$. (838) That is, *the Surfaces of Spheres are to each other as the Squares of their Diameters.*

843. From what has been said, it follows, that $D^3 ; DP :: D^2 : P$; that is, *any Cube* (D^3) *is to the Superficies of its inscribed Sphere* (DP) *as a Square* (D^2) *is to its inscribed Circle.*



844. To cube the Elliptic Spheroid ABDE. Here $BP = x$, $PM = y$, $BE = a$, $AD = b$, $p =$ Parameter. Then $yy = px - \frac{px^2}{a}$ (763.) Let the Radius of a Circle be to the Circumference, as $r : c$; then $r : c :: y : \frac{cy}{r} =$ Periphery of the Circle, whose Diameter is FM; then $\frac{cyy}{2r} =$ Area of the Circle; and so $\frac{cyy\dot{x}}{2r} = \frac{pc\dot{x}}{2r} - \frac{pcx^2\dot{x}}{2ra} =$ Fluxion of the Segment FBM; whose Fluent $\frac{pcx^2}{4r} - \frac{pcx^3}{6ra} =$ Solidity thereof. When $x = a$, then the said Fluent is $\frac{pca^2}{4r} - \frac{pca^2}{6ra} = \frac{pca^2}{12r} =$ Solidity of the *Spheroid*, as required.

845. Now because $a : b :: b : p = \frac{bb}{a}$ (766) and $b = 2r$; we have $\frac{pca^2}{12r} = \frac{cab}{6} =$ the Solidity of the Spheroid. And because

$\frac{cba}{6} = \frac{cb}{6} \times a$, and $\frac{cb}{4} \times a =$ Cylinder circumscribed about the Spheroid, 'tis evident, *the Spheroid is also* $\frac{2}{3}$ *of its circumscribing Cylinder*.

846. Hence in the Sphere $\left(\frac{cd^2}{6}\right)$ and Spheroid $\left(\frac{abc}{6}\right)$, when $d = b$, it is the Sphere : Spheroid :: $cd^2 : abc :: d : a ::$ Conjugate : Transverse Axis of the Ellipsoid.

847. Hence the Solidity of $\begin{cases} \text{the Cube} = 10000. \\ \text{the Cylinder} = 7854. \\ \text{the Sphere} = 5236. \\ \text{the Cone} = 2618. \end{cases}$

CHAP. XV.

The Nature *and* Genesis *of* LOGARITHMS, *deduced from the* HYPERBOLA, *the* MODULES *and* SCALES *of* LOGARITHMS; *the* FLUXIONS *of* LOGARITHMS, *and their Use.*

848. I Have already observed, that a Logarithm is only an Expression of the Place or Distance from Unity, which any given Number holds in a Series of geometrical Proportionals (see 147). I have also shewn how those Logarithms are investigated and expressed by *Numbers*. Also from the Nature and Quadrature of the *Hyperbola* it appears (784, 785, 786) that the Ratio of any Numbers to Unity, may also be expressed by hyperbolic Sectors or Spaces, and consequently they are also properly the Logarithms of such Numbers. And we shall now more fully prosecute and explain this useful Doctrine.

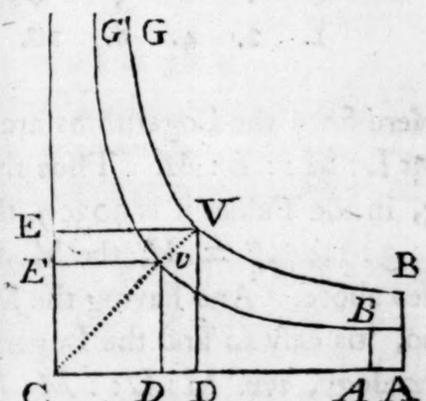
849. It has been shewn, that the Asymptotic Space ADVB $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, *&c.* is the Logarithm of the Ratio of CD to CA, that is, of $a + x$ to a, or of $1 + x$ to 1 (786). And because $\frac{\dot{x}}{1 + x}$ is the Fluxion of this Ratio, therefore it appears, *that the Fluxion of the Logarithm of any Number,*

$1 + x$, *is equal to the Fluxion* ($\dot{x}$) *of that Number divided by the Number itself.*

850. If we take any Number less than Unity, as C*A*; (See Fig. to Art. 829.) then *A* D $= x$, and the Logarithm of CD : C*A*, or of $1 : 1 - x$, is the Space *A B* V D $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$, &c. Whence the Logarithm of the Ratio of C A : C *A*, or of $1 + x$ to $1 - x$, the Space A B *B A* $= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7$, &c. = Sum of the other two Series, (Art. 849.)

851. Let A D = *A* D $= \frac{1}{10} = x$, then A C $= 1 + \frac{1}{10}$, and *A* C $= 1 - \frac{1}{10}$. Now half the Sum of the above two Series is $x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7$, &c. and half their Difference is $\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{6}x^6 + \frac{1}{8}x^8 +$, &c. which computed in Numbers make, 0,1003353477310; and 0,0050251679267; the *Sum* of these gives the greater Logarithm *A B* V D = 0.1053605156, which is the hyperbolical Logarithm of the Ratio of 1 to 0.9, or of 10 to 9. Also their Difference gives the lesser Logarithm A B V D = 0.09531018, which expresses the Ratio of 1.1 to 1, or 11 to 10. Lastly, the Sum of the two Series, *viz.* $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7$, &c. = 0.20066707 = *AB* B A, the Logarithm of the Ratio of 1,1 to 0,9, or 11 to 9.

852. By proceeding in this Manner, the Logarithm of the Ratio of any Number to Unity may by Degrees be found; and that of 10 to 1 will be found to be 2.30258509299, &c. which Number is therefore called the *hyberbolical Logarithm* of 10. But in the Logarithms of the Tables in common Use, the Logarithm of the same Ratio of 10 to 1 is 1,0000000, &c. which is called the *Tabular Logarithm* of 10.



853. Let there be two similar Hyperbolas BVG and *BVG*, *viz.* such *whose transverse and conjugalate Diameters are proportional*, or C V : C*V* : : C D : C *D*. And let C A : C D, C *A* : C *D* in each be the Ratio of 10 to 1. And let the Logarithm of this Ratio in the first, *viz.* A D V B = 2,3025851, &c. and the Logarithm of the same Ratio in the latter, *viz.* *ADVB* = 1,0000000. Then it will

will be $\mathrm{ADVB} : ADVB :: \mathrm{CDVE} : CDVE$. For put $\mathrm{CD} = a$, $\mathrm{AD} = x$; $CD = \mathrm{a}$, $AD = \mathrm{x}$; then will ADVB $= a^2 \times \overline{\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3}}$, &c. And $ADVB = \mathrm{a}^2 \times$ $\overline{\frac{\mathrm{x}}{\mathrm{a}} - \frac{\mathrm{x}^2}{2\mathrm{a}^2} - \frac{\mathrm{x}^3}{3\mathrm{a}^3}}$, &c. Now since $\mathrm{CA} : \mathrm{CD} :: CA : CD$, that is, $a : x :: \mathrm{a} : \mathrm{x}$; we have $a\mathrm{x} = \mathrm{a}x$; and so $\frac{x}{a} = \frac{\mathrm{x}}{\mathrm{a}}$; therefore $\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3}$, &c. $= \frac{\mathrm{x}}{\mathrm{a}} - \frac{\mathrm{x}^2}{2\mathrm{a}^2} - \frac{\mathrm{x}^3}{3\mathrm{a}^3}$, &c. and consequently it is $\mathrm{ADVB} : ADVB :: a^2 : \mathrm{a}^2 :: CDVE : CDVE : 2{,}3025851 : 1.0000000 :: 1 : 0.4342945$.

854. Now these Parallelograms CDVE, *CDVE*, are called the MODULES of the *Systems* or *Scales* of Logarithms, in each Hyperbola respectively; let these be called M and *M*; and L and *L* represent the Logarithms of any given Number; then $\mathrm{M} : M :: \mathrm{L} : L$; and so $\frac{M\,\mathrm{L}}{\mathrm{M}} = L$, put $\frac{M}{\mathrm{M}} = \mathrm{R} =$ 0,4342945, the Module of the common Tables. Then $\mathrm{RL} = L = 0{,}4342945\ \mathrm{L} =$ the *tabular Logarithm* of the given Number. And $\frac{\mathrm{M}\,L}{M} = \mathrm{L} = 2{,}3025851\ L =$ the Logarithm whose Module is M, or C D V E.

855. To illustrate this, let a Series of Numbers, and their Logarithms be as follow:

0	1	2	3	4	5	6, &c.	Logarithms.
1.	2.	4.	8.	16.	32.	64.	Numbers.

Here since the Logarithms are given, we can find the Module; for $\mathrm{L} : \mathrm{M} :: L : M$. Thus the Logarithm of 8 in this Series is 3, in the Tables it is 90309, therefore 0.90309 : 0.4342944819 :: 3 : 1,4427 = *M*, the Module of the Logarithms in the Series above. And having the Module of the Logarithm or Ratio, 'tis easy to find the Logarithm by the Reverse of the last Analogy, *viz.* $\mathrm{M} : \mathrm{L} :: M : L$. Thus 0.4343 : 0.90309 :: 1.4427 : 3 = the Logarithm of 8.

856.

856. Also the hyperbolical Logarithms are easily produced by multiplying those of the common Tables by 2,3025851, as in the following Examples of all the Numbers from 1 to 10. Thus

Table of Logarithms.		Hyperb. Logarithms.
1—0.0000000		1—0.0000000
2—0.3010300		2—0.6931472
3—0.4771213		3—1.0986123
4—0.6020600	Multipled by	4—1.3862943
5—0.6989700	2.3025851	5—1.6094379
6—0.7781513	produce.	6—1.7917594
7—0.8450980		7—1.9459101
8—0.9030900		8—2.0794415
9—0.9542425		9—2.1972241
10—1.0000000		10—2.3025851

857. We shall next shew how to find the Fluxion of Logarithms and the Powers; as also the Fluxions of Exponential Quantities, of any Degree. In order to find the Fluxion of the Logarithm of any Quantity, as x, or $x+1$, or $\overline{x+a}^m$, we shall denote the Logarithm by L, and its Fluxion by $\dot{L}$; and it is to be understood that L, and $\dot{L}$ respects those Quantities only to which they are immediately prefixed: Thus $L\,\overline{x+y}$ is the Logarithm of $\overline{x+y}$; but $L\,x+y$ is only the Logarithm of x added to the Quantity y.

858. Hence it will be easy to find the Fluxion of any Kind of Logarithm. For Example; $\dot{L}\,\overline{xx+yy} = \frac{2x\dot{x}+2y\dot{y}}{xx+yy}$. $\dot{L}\,\overline{ax^2+x^3} = \frac{2ax\dot{x}+3x^2\dot{x}}{ax^2+x^3} = \frac{2a\dot{x}+3x\dot{x}}{ax+x^2}$; and universally $\dot{L}\,\overline{z^m+x^n}^q = \frac{qmz^{m-1}\dot{z}+q^n x^{n-1}\dot{x}}{z^m+x^n}$. For put $z^m+x^n=y$, then will $\overline{z^m+x^n}^q = y^q$; and consequently the Fluxion of the Logarithm thereof will be $= q\dot{y}\times\frac{y^{q-1}}{y^q} = q\dot{y}\times\frac{1}{y} = \frac{q\dot{y}}{y}$; which will be found equal to the Fluxion above, by substituting the Values of y, and $\dot{y}$.

859.

859. The Fluxions of the Powers of Logarithms are found by the general Rules for the Fluxion of the Powers of flowing Quantities. Thus admit the Fluxion of the 2d, or *Square Power* of the Logarithm of x were required, or Fluxion of $\overline{\mathrm{L}\,x}^2$; we must first multiply by the Index of the Power, *viz.* 2, which makes $2\,\overline{\mathrm{L}\,x}^2$; then is the Index to be lessened by Unity, which makes $2\,\overline{\mathrm{L}\,x}^{2-1} = 2\,\mathrm{L}\,x$; lastly, we must multiply this into the Fluxion of the Root $\mathrm{L}\,x$, which is $\frac{\dot{x}}{x}$, and it will produce $2\,\mathrm{L}\,x \times \frac{\dot{x}}{x}$ the Fluxion of $\mathrm{L}^2\,x$, as required.

860. And in general the Fluxion of $\mathrm{L}^m\,x = m\,\mathrm{L}^{m-1}\,x \times \frac{\dot{x}}{x}$; and $\mathrm{L}^n\,\overline{x+a} = n\,\mathrm{L}^{n-1}\,\overline{x+a} \times \frac{\dot{x}}{x+a}$; and $\mathrm{L}^m\,\overline{x+a}^n = m\,\mathrm{L}^{m-1}\,\overline{x+a}^n \times \frac{n\,\dot{x}}{x+a}$. For $\mathrm{L}^m\,\overline{x+a}^n$ multiplied by the Index of the Power is $m\,\mathrm{L}^m\,\overline{x+a}^n$; and that with its Exponent m lessened by Unity, is $m\,\mathrm{L}^{m-1}\,\overline{x+a}^n$, and this being multiplied by the Fluxion of the Root $\mathrm{L}\,\overline{x+a}^n$, (which is $\frac{n\,\overline{x+a}^{n-1}\,\dot{x}}{\overline{x+a}^n} = \frac{n\,\dot{x}}{x+a}$,) will make $m\,\mathrm{L}^{m-1}\,\overline{x+a}^n \times \frac{n\,\dot{x}}{x+a}$ the same as above. And thus may the Fluxions of the Powers of Logarithms, however compounded with other Quantities, or Powers of Quantities, be found.

861. From what has been said, we may observe that the Fluxions of the Logarithms of the several Powers of any Quantity, as x, are in the same Ratio as the Exponents of those Powers, and consequently as the Logarithms themselves. Thus $\dot{\mathrm{L}}\,x^m$ and $\dot{\mathrm{L}}\,x^n$ are $\frac{m\,\dot{x}\,x^{m-1}}{x^m} = \frac{m\,\dot{x}}{x}$ and $\frac{n\,\dot{x}\,x^{n-1}}{x^n} = \frac{n\,\dot{x}}{x}$; but $\frac{m\,\dot{x}}{x} : \frac{n\,\dot{x}}{x} :: m : n$. Which proves the Truth of what I said.

862. We have hitherto supposed the Indexes, or Exponents of Powers *determinate*, or *invariable*, but in some Cases we shall find, that they also are variable or flowing Quantities, and have accordingly

accordingly their Indexes or Exponents. Now the Fluxions of those Exponential Quantities, whose Indexes are variable, are found by Means of the Logarithms very easily, as in the following Examples.

863. Let it be required to find the Fluxion of a^x, where a is a constant Quantity, and its Index x a flowing one: Put $a^x = z$, then will $\dot{z}$ be equal to the Fluxion of a^x. Because $a^x = z$, if we take the Logarithms on both Sides, we shall have $L\,a^x = L\,z$; and *since the Logarithm of the Root of any Power multiplied by the Index of the Power, is the Logarithm of that Power*, therefore $x\,L\,a = L\,a^x$, and so $x\,L\,a = L\,z$; and taking the Fluxions, we have $\dot{x}\,L\,a = \frac{\dot{z}}{z}$, and therefore $z\,\dot{x}\,L\,a = a^x\,L\,a\,\dot{x} = \dot{z}$; and consequently $a^x\,L\,a\,\dot{x}$ is the Fluxion of a^x, as required.

864. Again, suppose it required to find the Fluxion of z^x, where both z and x are flowing Quantities, and make $z^x = v$, then will $L\,z^x = L\,v = x\,L\,z$. The Fluxions of these will be $\frac{\dot{v}}{v} = \dot{x}\,L\,z + x\frac{\dot{z}}{z} = \frac{\dot{v}}{z^x}$ (because $v = z^x$,) and by reducing the Equation, we have $z^x\,\dot{x}\,L\,z + z^{x-1}\,x\,\dot{z} = \dot{v}$; which therefore is the Fluxion sought. *Note*, $x\frac{\dot{z}}{z} = x\,\dot{z} \times \frac{1}{z}$, and $\frac{1}{z} \times z^x = \frac{z^x}{z} = z^{x-1}$; *and therefore* $x\frac{\dot{z}}{z} \times z^x = z^{x-1}\,x\,\dot{z}$, *as in the latter Part of the Fluxion.*

865. Lastly, let the Quantity z^{y^x} be proposed, to find its Fluxion. Let $z^{y^x} = v$; then $L\,z^{y^x} = L\,v$, or $y^x\,L\,z = L\,v$; and taking the Fluxions, we have the Fluxion of $y^x = y^x\,L\,y\,\dot{x} + y^{x-1}\,x\,\dot{y}$ (864) which must be multiplied into $L\,z$, which makes $L\,z \times y^x\,L\,y\,\dot{x} + L\,z \times y^{x-1}\,x\,\dot{y}$; to which must be added the Fluxion of $L\,z \times y^x$, or $y^x\frac{\dot{z}}{z}$; and the Sum will be equal to the Fluxion of $L\,v$, or $\frac{\dot{v}}{v}$; thus $L\,z\,y^x\,L\,y\,\dot{x} + L\,z\,y^{x-1}\,x\,\dot{y} + y^x\frac{\dot{z}}{z} = \frac{\dot{v}}{v} = \frac{\dot{v}}{z^{y^x}}$. And therefore multiplying by z^{y^x}, we have

z^{y^x}

$$z^{y^x} \mathrm{L}z \times y^x \mathrm{L}y\,\dot{x} + z^{y^x} \mathrm{L}z \times y^{x-1} x\,\dot{y} + y^x\,\dot{z}\,z^{y^x-1} = \dot{v};$$

and is therefore the Fluxion of the Expression z^{y^x}, as required.

CHAP. XVI.

The METHOD *of finding* FLUENTS *by the Measures of* RATIOS *and* ANGLES, *by the Help of* TABLES *of* LOGARITHMS; *and of Natural* SINES, TANGENTS, *&c.*

866. AS the Method of finding *Fluents* of given Fluxions is the most necessary, and yet the most abstruse Part of the *new Mathesis*, it is no Wonder if it has exercised the inventive Faculty of the greatest Geniuses of the last and present Age. And since in the *Geometry* of *Curve Lines*, and especially in the Solution of *Physical Problems*, (which are of the highest Importance in the Sciences), Cases will often occur, which make it necessary to have Recourse to *Infinite Series*, for procuring the Fluents required, and this often proves a laborious and disagreeable Task; and which of Course the Student would be glad to avoid or exchange for some other more easy and practical Method.

867. Now there are several Forms of Fluxions whose Fluents may be readily obtained in the Measure of *a Ratio*, and also of the *Arch of a Circle*; and consequently, as all Ratios of Numbers to Unity, are already calculated in the common Tables of Logarithms of Numbers; and any Arch of a Circle is known from its *Right Sine*, *Versed Sine*, *Tangent*, or *Secant*, calculated in the Trigonometrical Canon, it follows, those Tables will greatly facilitate the Business of finding Fluents; and the Method of using them for that Purpose, we shall next explain and illustrate somewhat farther than has been yet done for the Capacity of a *Learner*.

868. For this Purpose he must recollect, *That the Fluxion of a Hyperbolic Logarithm of any Number, is ever expressed by the Fluxion of that Number, divided by the Number itself* (849). And hence

hence will arise the four following *Logarithmic* Forms of Fluxions, whose Fluents may be had from a Table of Logarithms, which would otherwise require an *infinite Series* to determine them, *viz.*

869. The Fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ will be expressed by the hyperbolical Logarithm of $x + \sqrt{x^2 \pm a^2}$: For the Fluxion of ($x + \sqrt{x^2 \pm a^2}$) the Number itself, being $\dot{x} + \frac{x\dot{x}}{\sqrt{x^2 \pm a^2}}$ $= \frac{\dot{x}\sqrt{x^2 \pm a^2} + x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}} \times \overline{\sqrt{x^2 \pm a^2} + x}$, this last Quantity, divided by that Number, gives $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$; the very Fluxion first proposed; which is the *first Form.*

870. It also appears that the Fluent of $\frac{\dot{x}}{\sqrt{2ax + x^2}}$ will be truly expounded by the hyperbolical Logarithm of $a + x + \sqrt{2ax + x^2}$: Because the Fluxion of the Number ($a + x + \sqrt{2ax + x^2}$) is here $= \dot{x} + \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}} = \frac{\dot{x}}{\sqrt{2ax + xx}}$ $\times \overline{\sqrt{2ax + xx} + a + x}$; which divided by that Number produces $\frac{\dot{x}}{\sqrt{2ax + xx}}$; which is the *second Form.*

871. Likewise the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ will be represented by the hyperbolical Logarithm of $\frac{a+x}{a-x}$: Because the Fluxion of $\frac{a+x}{a-x}$, being $\frac{\dot{x} \times \overline{a - x} + \dot{x} \times \overline{a + x}}{\overline{a - x}|^2} = \frac{2a\dot{x}}{\overline{a-x}|^2}$, if the same be therefore divided by $\frac{a+x}{a-x}$, we shall have $\frac{2a\dot{x}}{\overline{a-x}|^2} \times \frac{a-x}{a+x}$ $= \frac{2a\dot{x}}{\overline{a-x} \times \overline{a+x}} = \frac{2a\dot{x}}{a^2 - x^2}$; which is the *third Form.*

872. Lastly, the Fluent of $\frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$ will be denoted by the

the hyperbolical Logarithm of $\frac{a-\sqrt{a^2 \pm x^2}}{a+\sqrt{a^2 \pm x^2}}$; for here the Fluxion of the Number is $\frac{\mp x\dot{x}}{\sqrt{a^2 \pm x^2}} \times \frac{\overline{a+\sqrt{a^2 \pm x^2}}}{\overline{a+\sqrt{a^2 \pm x^2}}|^2}$

$$\mp \frac{x\dot{x}}{\sqrt{a^2 \pm x^2}} \times \frac{\overline{a-\sqrt{a^2 \pm x^2}}}{\overline{a+\sqrt{a^2 \pm x^2}}|^2} = \frac{\mp 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times \overline{a+\sqrt{a^2 \pm x^2}}|^2};$$

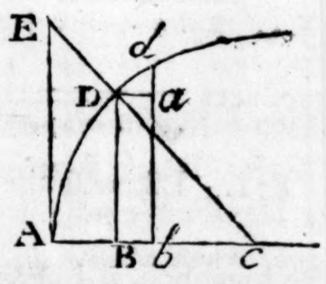
which divided by $\frac{a-\sqrt{a^2 \pm x^2}}{a+\sqrt{a^2 \pm x^2}}$ gives $\frac{\mp 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times \overline{a+\sqrt{a^2 \pm x^2}}|^2}$

$$\times \frac{a+\sqrt{a^2 \pm x^2}}{a-\sqrt{a^2 \pm x^2}} = \frac{\pm 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times \overline{a+\sqrt{a^2 \pm x^2}} \times \overline{a-\sqrt{a^2 \pm x^2}}}$$

$= \frac{\mp 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times \mp x^2} = \frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$, the Fluxion proposed, whch is the *fourth Form.*

These four are the principal Forms of Fluxions; whose Fluents may be found from a Table of Logarithms of the hyperbolic Kind: Which Table, upon Occasion, may be easily supplied by a Table of the common Form, as we have shewn (856).

873. Four other Forms of Fluxions similar to these, express the Fluxion of the Arch of a Circle in Terms of the Sine, Versed Sine, Tangent and Secant of that Arch. For let A D be the Arch of a Circle, whose Center is C, its Sine B D, Versed Sine A B, Tangent A E, and Secant C E, let *b d*, be drawn indefiniently near to B D, and draw D *a*, parallel to A C, then since the very small Arch D *d*, may be esteemed a right Line, the small fluxionary Triangle D *a d* will be similar to the Triangle D B C or A E C, for the Angle at *a*, and B is a right one in each; also, if from the Right Angles C D *d*, and B D *a*, we take the common Angle C D *a*, there will remain the Angle *d* D *a* = B D C, and the Sides in each, *viz.* *d a*, and D *a* being parallel to the Sides B D and B C, they will be proportional, and the Triangles similar by (621, 657).



874.

874. Put the Radius A C $= a$, the Verſed Sine A B $= x$, the Sine B D $= y$, the Tangent A E $= t$, the Secant C E $= s$, and the Arch A D $= z$, then in the fluxionary Triangle D $a\,d$, we have D $a = \dot{x}$, $a\,d = \dot{y}$, and D $d = \dot{z}$, whence we have the following Analogies, C B : C D : : D a : D d, that is, $\sqrt{a^2 - y^2} : a :: \dot{y} : \dot{z} = \frac{a\dot{y}}{\sqrt{a^2 - y^2}}$, which correſponds to the firſt Form in (869).

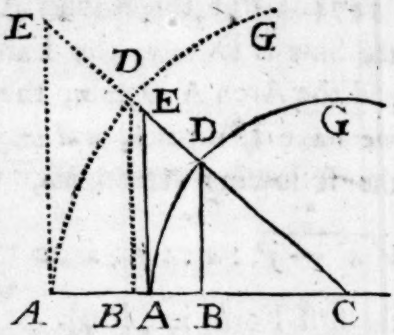
875. Again, we have the following Analogy in Terms of the Verſed Sine, as B D : C D : : D a : D d, that is, $y : a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{y} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ by (636). This Form is analogous to the ſecond Form in (870.)

876. The Fluxion of the ſame Arch, expreſſed in Terms of the Tangent, we have already ſhewn in (823,) is $\dot{x} = \frac{a^2\dot{t}}{a^2 + t^2}$, which is ſimilar to the third Form in (871).

877. Laſtly, by ſimilar Triangles CBD, CAE, we have CE : CA : : CD : CB; that is, $s : a :: a : \frac{a^2}{s}$; hence A B $= x = a - \frac{a^2}{s}$, whoſe Fluxion is $\dot{x} = \frac{a^2\dot{s}}{s^2}$; but A E : CE : : D a : D d; that is, $\sqrt{s^2 - a^2} : s :: \frac{a^2\dot{s}}{s^2} : \dot{z} = \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}}$, which is like the fourth Form in (872).

878. Theſe Forms differ in nothing from the former, but the Signs and conſtant Quantities, by which it will be eaſy to know when the Fluent is to be ſought for in the Meaſure of a Ratio, or of an Angle; that is from a *Table of Logarithms*, or the Trigonometrical *Canon of Sines*, *Tangents*, &c. But as in the above Forms, the Radius is expreſſed by (a), and the Radius of the Table by Unity, or 1.000000; therefore to accomodate them to the Tables in uſe, we take the Expreſſion of the *Sine*, *Tangent*, &c. ſuch as pertain to a Circle, whoſe Radius $= 1$, in the following Manner,

879. With the Radius C$A = a$, describe the Circle ADG; and with the Radius C A $= 1$, describe the Circle A D G; draw the indefinite Line CE, and from the Points D, D, let fall the Perpendiculars D B, DB; and at A and A raise the Perpendiculars AE, AE. Then by similar Triangles C A : CA : : AD : AD; whence (putting AD $=$ D, and $AD =$ A), we have $1 : a :: \mathrm{D} : a\mathrm{D} = \mathrm{A}$. Again, put D B $=$ y, and $DB = y$; then C D : CD : : D B : DB; that is, $1 : a :: \mathrm{y} : a\mathrm{y} = y$. In like Manner, by making A E $=$ t, and $AE = t$; we have $1 : a :: \mathrm{t} : a\mathrm{t} = t$; and if C E $=$ s, and C$E = s$; then $1 : a :: \mathrm{s} : a\mathrm{s} = s$. Lastly, putting A B $=$ x, and $AB = x$, then BC $= 1 - \mathrm{x}$, and BC $= a - x$, but it is $1 : a :: 1 - \mathrm{x} : a - x$, and consequently $1 : a :: \mathrm{x} : x = a\mathrm{x}$.



880. Hence the Tabular Sines, Tangents, *&c.* expressed in Terms of the above fluxionary Forms, will be $\mathrm{y} = \frac{y}{a}$, $\mathrm{t} = \frac{t}{a}$, $\mathrm{s} = \frac{s}{a}$, and $\mathrm{x} = \frac{x}{a}$. And therefore since the Fluxion of the Arch AD ($=$ A $= a \times$ D) is $\frac{a\dot{y}}{\sqrt{a^2 - y^2}}$ (874); the Fluxion of the Arch A D $=$ D, will be only $\frac{\dot{y}}{\sqrt{a^2 - y^2}}$; and consequently it will follow,

That the Fluent of	is equal to the Arch, whose Radius is Unity, and	is
$\frac{\dot{y}}{\sqrt{a^2 - y^2}}$	Right Sine	$\frac{y}{a}$
$\frac{\dot{x}}{\sqrt{2ax - x^2}}$	Versed Sine	$\frac{x}{a}$
$\frac{a\dot{t}}{a^2 + t^2}$	Tangent	$\frac{t}{a}$
$\frac{a\dot{s}}{s\sqrt{s^2 - a^2}}$	Secant	$\frac{s}{a}$

881. But to apply this Doctrine more generally to Use, with regard to finding the Fluent by Logarithms or Trigonometrical

metrical Numbers, it is to be obſerved, that if the Terms of the Ratio conſiſt of ſeveral Quantities, as $\frac{A+B}{C}$, and any one of them, as A, be conſtant or given, and the Relation of the Reſt depend upon their Connection with it, then the ſaid Quantity A may be aſſumed for the *Module* of that Ratio; and then (according to the Method invented by the late learned Mr. *Cotes*) it may be expreſſed in this Manner, $A \left| \frac{A+B}{C} \right.$; *viz. the Ratio of* A + B *to* C, *in that Scale of Proportionals, the Module of whoſe Logarithms is* A.

882. Thus, ſuppoſe it were required to find the Fluent of the Fluxion $\frac{6a\dot{x}}{b \times \overline{a^2 - x^2}}$; this you will obſerve conſiſts of a conſtant and a fluxionary Part, *viz.* $\frac{3}{b} \times \frac{2a\dot{x}}{a^2 - x^2}$, and the Fluent of the Latter is $\frac{a+x}{a-x}$ (871), which, (as it is a Ratio, and the Part (a) invariable), may be thus expreſſed, $a \left| \frac{a+x}{a-x} \right.$; the Logarithm, or Meaſure of this Ratio is a L R *, which is the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$; therefore $\frac{3a}{b} \left| \frac{a+x}{a-x} \right.$, or $\frac{3a}{b} \times$ L R, is the Fluent of $\frac{6a\dot{x}}{b \times \overline{a^2 - x^2}}$, as required.

883, To purſue the *Coteſian Method* yet farther, let the Right angled Triangle A C E be propoſed, See Fig. to (873) make A C = R = *Radius*; then will A E = T = *Tangent*, and C E = S = *Secant*, of the Angle A C E. Then if it be propoſed to find the Meaſure of the Ratio of the Sum of the Legs to the Hypothenuſe, the Baſe A C being the Module, the

* *N. B.* A Miſtake in Art. 854, paſſed (thro' Inadvertency) uncorrected.—The latter Part of that Article is to be read thus; M : M :: (0.4342945 : 1 ::) L : L; and ſo $\frac{M\,L}{M} = L$; put $\frac{M}{M} \left(= \frac{1}{0{,}4342945}\right)$ = R = 2,3025851; then will R L = L = *Hyperbolical Logarithm*, or Meaſure of a Ratio in the Scale, whoſe Module is 1. Alſo M and L may repreſent the Module and Logarithm of any other Scale beſides that of the Hyperbola, and accordingly R may be determined.

the Expression will be thus; $R \left| \frac{R+T}{S} \right.$, or if $R = a$, $T = y$, then will $S = \sqrt{aa + yy}$; and the Ratio stands thus; $a \left| \frac{a+y}{\sqrt{aa+yy}} \right.$. If now $R = 4$, $T = 3$, $S = 5$; then will the Ratio in Numbers be $4 \left| \frac{4+3}{5} \right. = 4 \left| \frac{7}{5} \right.$; the Measure of which is found as before, thus; $0.4343 : 0.14613$ ($=$ Table of Logarithms of $\frac{7}{5}$) $: : 4 : 2.6917 =$ Logarithm or Measure of the Ratio of $\frac{7}{5}$ to the Modulus 4.

884. If the Fluent cannot be found in the Expression of a Ratio, (as it sometimes happens) then we can often obtain it by the Measure of an Arch of a Circle, as A D, (See Fig. to 873), or the given Angle A C E; for since the Circumference of a Circle is to its Diameter, as 3.14159, &c. to 1, if we say, as 3.14159, &c. : 1 : : 180 Degrees : 57.2957795, &c. $= 57° \ 17' \ 44''$ = the Degrees in the Arch of a Circle equal to Radius; this Number therefore is called the *Trigonometrical Modulus*, and is represented by $K = 57.2957795$, &c. and if $\frac{1}{57.2957795} = 0.01745329252 = \frac{1}{K} = k$, then is k called the *Reciprocal Modulus*.

885. Suppose now any Arch of a Circle $AD = D$; then it will be, *As the Radius* A C *in Degrees to the Arch* A D *in Degrees, so is the Radius* A C *in Numbers to the Arch* A D *in Numbers*; that is, as $K : D : : R : \frac{R}{K} D = R k D =$ the Arch A D in Numbers. For Example, If $R = 1$, and $AD = D = 30°$, then $R k D = 1 \times 0.0174530 \times 30 = 0.523598$, &c. = the Measure of the Arch A D in Numbers.

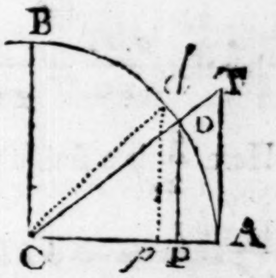
886. In order to find the Fluents in the Measure of a Ratio or Angle, in the Manner as has been taught, Mr. *Cotes* invented a great many *general Forms of fluxionary Expressions* (to which others are to be reduced) and disposed in Tables, the various Fluents pertaining thereto, according to the different Values of the Exponent of the flowing Quantity, affirmative or negative; which the Reader will find at large in his learned Treatise, entituled *Harmonia Mensurarum*.

887.

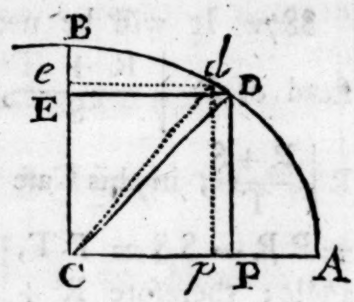
887. It will be neceſſary to obſerve here, that, if inſtead of $R \left| \frac{R+T}{S} \right.$, we ſhould meet with the Ratio $R \left| \frac{R+S}{T} \right.$, in this Caſe $\frac{R+S}{T} = \frac{T}{S-R}$; for $\overline{R+S} \times \overline{R-S} = RR - SS = TT$, by the Property of a right angled Triangle; therefore $R+S : T :: T : R-S$; and conſequently $\frac{R+S}{T} = \frac{T}{S-R}$; and ſo $R \left| \frac{R+S}{T} \right. = R \left| \frac{T}{R-S} \right.$; which to know will be of uſe hereafter.

888. Alſo it will be proper to obſerve, that the negative Expreſſion of the Logarithm of a Ratio, as $-R \left| \frac{R+T}{S} \right.$, is made Affirmative by inverting the Ratio, as thus $+R \left| \frac{S}{R+T} \right.$. For $\frac{a^2}{a^5} = a^{-3}$; but $\frac{a^5}{a^2} = a^3$. Or if $-\frac{a}{b} = s$, then $-s = +\frac{b}{a}$. I ſhall illuſtrate this Doctrine by a Solution of the following Problems.

889. *To ſquare the* Sector *of a Circle* CAD. Let $CA = a$, $AP = x$, and $PM = y$, then by (874), we find the Fluxion of the Arch $AD = \frac{a\dot{y}}{\sqrt{aa-yy}}$ $= Dd$; which multiplied by $\frac{a}{2}$ is $\frac{aa\dot{y}}{2\sqrt{aa-yy}} = dCD$, which ſmall Sector is the Fluxion of the Sector ACD. Now $\frac{a}{2}$ is a ſtable Quantity, and the other Part $\frac{a\dot{y}}{\sqrt{aa-yy}}$ being the Fluxion of the Arch AM, its Fluent will be found in the Meaſure of the Angle ACD, *viz.* it will be $= akD$ (885); and if $AD = D = 30°$, the Length $AD = 0.523598$, &c. which multiplied by $\frac{a}{2}$, will give 0.261799, &c. $=$ Area of the Sector ACD, when $a = 1$. (See Art. 828.)



890.



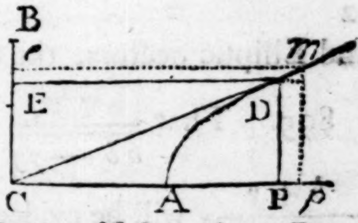
890. *To square the* Sector *of an Ellipsis* ACD. Let AC $= a$, CB $= b$, AP $= x$, CP $=$ ED $= v$, PD $= y$. 'Tis evident the Fluxion of the Space CADE is E$d =$ $v\dot{y}$; also the Triangle CDE $= \frac{vy}{2}$, whose Fluxion is $\frac{v\dot{y} + y\dot{v}}{2}$, which subtracted from the Fluxion $v\dot{y}$, leaves $\frac{\dot{y}v - \dot{v}y}{2} =$ to D C d, the Fluxion of the Sector ACD. And since from the Equation of the Curve $\frac{aa}{bb} yy = aa - vv$, we have in Fluxions, $\frac{aa}{bb} y\dot{y} = - v\dot{v}$, or $\frac{aa}{bbv} y\dot{y} = - \dot{v}$; if we substitute this Value of $\dot{v}$ in the Expression $\frac{\dot{y}v - \dot{v}y}{2}$, we shall have $\frac{\dot{y}}{2v} \times \overline{vv + \frac{aa}{bb} yy} = \frac{aa\dot{y}}{2v}$, because $vv + \frac{aa}{bb} yy = aa$; also, because it is $v = \frac{a}{b}\sqrt{bb - yy}$, therefore $\frac{a.a\dot{y}}{2v} = \frac{ab\dot{y}}{2\sqrt{bb - yy}} =$ Fluxion of the Sector CAD. Here $\frac{a}{2}$ is a stable Quantity, and the Fluent of the other Part $\frac{b\dot{y}}{\sqrt{bb - yy}}$ is bkD, as shewn in the last Article; therefore $\frac{ab}{2} k$D $=$ Area of the Sector ACD, as required.

891. *To square the* Sector *of an Hyperbola* ACD. By the very same Method as in the Ellipsis, we get the Fluxion $\frac{ab\dot{y}}{2\sqrt{bb + y}}$, for the hyperbolical Sector ACD, which differs only in the Sign $+$ of the Quantity yy, from the other. Here the constant Part $\frac{a}{2}$ is the same as before, but the fluxionary Part $\frac{b\dot{y}}{\sqrt{bb + yy}}$ is different, or is not the Fluxion of an Arch of

or Angle; and therefore its Fluent must be sought in the Measure of a Ratio, by reducing it to one of the Forms in the Tables of Mr. *Cotes*. The Form which it agrees with, is the Sixth, *viz.* $\frac{d\dot{z}\, z^{rn+\frac{1}{2}n-1}}{\sqrt{e+fz^n}}$ to which the Fluxion $\frac{ab\dot{y}}{2\sqrt{bb+yy}}$ is reducible by the following Substitutions,

$$\textit{Viz.}\left\{\begin{array}{l} d=\frac{ab}{2} \\ z=y \\ r=o \\ n=2 \\ e=bb \\ f=1 \end{array}\right\}\text{Also at the Bottom of the Table you find}\left\{\begin{array}{l} R=\sqrt{f}=1. \\ T=\sqrt{\frac{e+fz^n}{z^n}}=\frac{1}{y}\sqrt{bb+yy} \\ S=\sqrt{\frac{e}{z^n}}=\frac{b}{y}. \\ \text{Also } \frac{2}{nf}dR=\frac{ab}{2}. \end{array}\right.$$

Therefore in the Table the Fluent which stands against $r=o$, is $\frac{2}{nf}dR \,\Big|\, \frac{R+T}{S}=\frac{ab}{2} \,\Big|\, \frac{y+\sqrt{bb+yy}}{b}=\frac{ab}{2}$ L R. Now since b, and y are given, the Logarithm L of the Ratio of $y+\sqrt{bb+yy}$ to b, will be given also; whence the Fluent $\frac{ab}{2}$ L R may easily be computed for the Area of Sector A C D. (See Art. 882.)



892. The Reason why the Fluent of the Fluxions of the *Circular and Elliptic Sectors* were not found in the Measure of a Ratio, is evident, because there f was a negative Quantity, or $f=-1$, and so it would have been $R=\sqrt{-1}$, which is impossible, there being no such Thing as the Square Root of a negative Quantity.

893. If the Tangent A T had been assumed for the variable Quantity, instead of the Sine P D; then the Fluxion of the *Circular Sector* would have been $\frac{aa\dot{y}}{2aa+2yy}$; for the *Elliptic Sector*, $\frac{abb\dot{y}}{2bb+2yy}$; and for the Hyperbolical Sector

$\frac{abb\dot{y}}{2bb-2yy}$. The Form in Mr. *Cotes*'s Table for this, is the Second, *viz.* $\frac{d\dot{z}z^{rn+\frac{1}{2}n-1}}{e+fz^n}$;

$$\text{Now putting}\left\{\begin{array}{l} d=abb \\ z=y \\ r=o \\ n=2 \\ e=2bb \\ f=-2 \end{array}\right\}\text{Then}\left\{\begin{array}{l} \mathrm{R}=\sqrt{\frac{-e}{f}}=b. \\ \mathrm{T}=z^{\frac{1}{2}n}=y. \\ \mathrm{S}=\sqrt{\frac{e+fz^n}{f}}=\sqrt{bb-yy} \\ \text{And, } \frac{2}{ne}d\mathrm{R}=\frac{1}{2}ab. \end{array}\right.$$

The Fluent against $r=o$ is $\frac{2}{ne}d\mathrm{R}\left|\frac{\mathrm{R}+\mathrm{T}}{\mathrm{S}}\right.=\frac{1}{2}ab$ $\left|\frac{b+y}{\sqrt{bb-yy}}\right.=\frac{1}{2}ab\mathrm{LR}$; which will give the Area of the Sector A C D, in the Hyperbola, the same as before; but it will not be the Fluent or Area of the Circle or Ellipsis, because there f is Affirmative, or $+f$, and therefore $\mathrm{R}=\sqrt{\frac{-e}{f}}$ $=-b$, which is impossible.

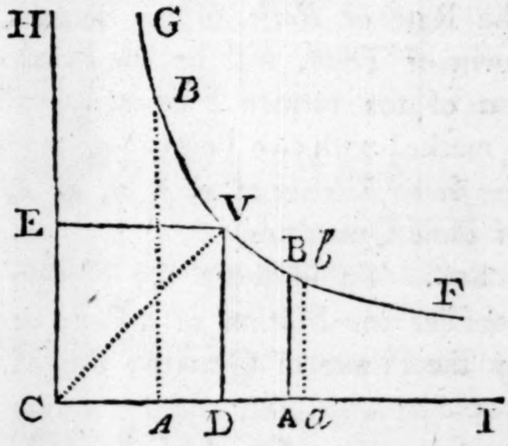
894. But since $\frac{aa\dot{y}}{aa+yy}$, or $\frac{bb\dot{y}}{bb+yy}$ are the Fluxions of an Arch, their Fluents will be $ak\mathrm{D}$, and $bk\mathrm{D}$, and therefore $\frac{aa}{2}k\mathrm{D}$ and $\frac{ab}{2}k\mathrm{D}$ will be the Fluents or Areas of the Circular and Elliptic Sectors, the same as before.

895. That $\frac{b\dot{y}}{bb-yy}$ is the Fluxion of the Logarithm of $\frac{b+y}{\sqrt{bb-yy}}$ will be evident, if we consider that the Fluxion of the Logarithm of $b+y$ is $\frac{\dot{y}}{b+y}=\frac{b\dot{y}-y\dot{y}}{bb-yy}$, and the Fluxion of the Logarithm of $\sqrt{bb-yy}=\frac{-y\dot{y}}{bb-yy}$. Now since the Fluxions of Logarithms have the same Properties with the Logarithms themselves (861), if we subtract $\frac{-y\dot{y}}{bb-yy}$ from $\frac{b\dot{y}-y\dot{y}}{bb-yy}$, there will remain $\frac{b\dot{y}}{bb-yy}=$

Fluxion

Fluxion of the Logarithm of $\frac{b+y}{\sqrt{bb-yy}}$, which was to be shewn.

896. It will be worth while to shew how easily the Fluent, or asymptotic Space ADVB is obtained by this Method from the Fluxion $\dot{x}y = \frac{aa\dot{x}}{a+x}$, (See Art. 829), which comes under Mr. *Cotes*'s *first Form, viz.* $\frac{d\dot{z}z^{rn-1}}{e+fz^n}$,

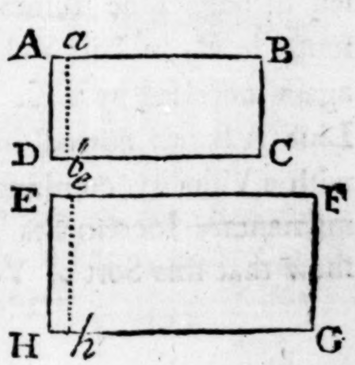


for putting $d=aa$, $z=x$, $n=1$, $r=1$, $e=a$, $f=1$; we have (against $r=1$,) the Fluent $\frac{d}{nf}\Big|\frac{e+fz^n}{e} = aa\Big|\frac{a+x}{a}$ $= CD^2\Big|\frac{CD+AD}{CD} = 1\Big|\frac{1+x}{1} = LR = 2.302585L$ $=$ Area of the Space ADVB, which therefore is given from the Logarithm of the Number $1+x$ in the common Tables. Hence it is again evident, that the hyperbolic Space ADVB is the Logarithm of $1+x$.

CHAP. XVII.

The Nature of SECOND, THIRD, *&c.* FLUXIONS *explained.*

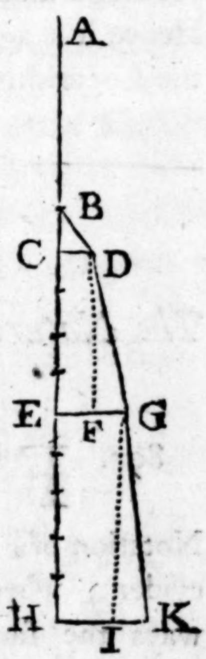
897. FROM what we have said of the Nature and Notation of Fluxions in general, it is evident, when the Velocity is always the same, the fluent Quantities AB and EF can have but one Sort of Fluxions Aa and Ee, which will be to each other, as the Velo-



 cities

cities of the Fluents reſpectively. But if the Velocity with which the Fluents A B and E F increaſe or decreaſe be not conſtant, but variable every Moment, then may this Velocity itſelf be conſidered as a *Fluent*, or *flowing Quantity*; and the Rate or *Ratio* of its Increaſe or Decreaſe each Moment of Time, will be the Fluxion thereof. And this Fluxion of the former Fluxion is called a *ſecond* Fluxion, and is marked with two Points over the ſame Letter; thus $\ddot{x}, \ddot{y}, \ddot{z}$ are the *ſecond Fluxions* of x, y, z, as $\dot{x}$, $\dot{y}$, $\dot{z}$, are the *firſt Fluxions* of thoſe Quantities.

898. To illuſtrate the Notion of *ſecond Fluxions*, let us conſider the Motion of a Body deſcending by its Weight, or by the Power of Gravity; the Motion reſulting from hence would be *accelerated*, and therefore the Velocity continually increaſing, (ſuppoſing the Fall of the Body to be in *Vacuo*) conſequently the Velocity of the Body the firſt Moment, (which is the *firſt Fluxion*) would be augmented the ſecond Moment, by the continual Action of Gravity, and this Augmentation will be the *ſecond Fluxion*, or *Fluxion of the firſt Fluxion*; and ſince this Acceleration or Increaſe of Velocity is regular and uniform, or proportional to the Times, 'tis plain this Sort of Motion will admit of nothing beyond *ſecond Fluxions*.

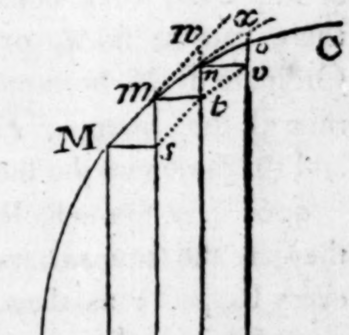


899. *For Example*; ſuppoſe from the loweſt Point B of the Line A B, an heavy Body begins to deſcend, and at the End of the firſt Moment it arrives to C, let the Velocity it then has, be expreſſed by the Line C D. At the End of the ſecond Moment it will have paſſed through four Times that Space, and be found in E, its Velocity then will be E G, increaſed by F G. At the End of the third Moment, it will have paſſed through nine Times the firſt Space, and be found in H, and its Velocity will then be H K, again increaſed by I K. In all this Time the Line A B has flowed into the Line A H, but with a Velocity continually increaſing; and its momentary Increments F G, I K being equal, ſhew that this Sort of Velocity admits of no further Variation,

and

and therefore of *Second Fluxions* only, or rather of *Fluxion*, in the abſolute Senſe; for I ſuppoſe I need not tell the Reader, that $\ddot{x}$ is but the ſame Fluxion of $\dot{x}$, as $\dot{x}$ is of x; and that $\ddot{x}$ has no Relation to the Fluxion of the Line A B, or x, but to the Velocity C D, or $\dot{x}$. The Want of this Hint in other Writers has left the young Student very often in the utmoſt Ambiguity and Confuſion.

900. The Fluxions of ſecond Fluxions are called *third Fluxions*; and are thus marked with three Points $\dddot{x}, \dddot{y}, \dddot{z}$. Theſe take Place in the Motion of an heavy Body deſcending by its Gravity at a great Diſtance above the Surface of the Earth; for in ſuch a Caſe, its Weight, or Power of Gravity increaſes continually, as the Squares of the Diſtances from the Earth's Center decreaſes: And therefore the momentary Increments F G of the firſt Velocity, (which before in ſmall Diſtances, near the Surface of the Earth, were eſteemed equal) will now receive a continual Increaſe, or Acceleration themſelves; and hence *third Fluxions* will ariſe. And ſince the nearer the Body approaches the Earth, the greater will be its Weight, theſe Accelerations will be conſtantly increaſed; and hence every Moment will produce a new Order of Fluxions, or *Third*, *Fourth*, *Fifth*, &c. *Fluxions*, will be generated *in Infinitum*, or ſo long as the Motion continues, as is eaſy to conceive.



901. What has been already ſaid, in regard to *Second* Fluxions, may be farther illuſtrated and explained, by conſidering the Ordinate P M, as moving along the Axis A B, of the Curve A M C, from A towards B: Let A P $= x$, P M $= y$, and A M $= z$.

902. If now in an indefinite ſmall Particle of Time, the Ordinate P M moves from that Situation to another $p\,m$; then, drawing M s parallel to A B, we ſhall eaſily obſerve, that $s\,m$ will be the Velocity, or Fluxion of P M, and M m the Fluxion of the Arch A M, and alſo that M s is the Fluxion of the Abſciſs A P, in the ſame Time; that is, M $s = \dot{x}$, $m\,s = \dot{y}$, and M m $= \dot{z}$.

903.

903. If in the next Moment the Ordinate move to $q\,n$; draw $m\,t$ parallel to AB, join $s\,t$, and draw parallel thereto $m\,w$, to interſect the Ordinate $q\,n$ in w; then will the Velocity with which the ſaid Ordinate decreaſeth in the ſecond Moment be $n\,t$, and the Difference between $m\,s$ and $n\,t$, or $w\,t$ and $n\,t$, *viz.* $w\,n$, will be the Variation of the Velocity, or Fluxion $m\,s$, that is, it will be the *Second Fluxion* of P M; and ſo $w\,n = \ddot{y}$.

904. Again, in the third Moment, ſuppoſe the Ordinate arrives to the Situation $r\,o$, draw $n\,v$ parallel to A B; join $t\,v$, to which draw $n\,x$ parallel, meeting the Ordinate $r\,o$ in x; then will $v\,o$ be its Fluxion the third Moment, which is leſs than that in the Second, by the Quantity $x\,o$, (becauſe $n\,t = x\,v$) therefore the Difference between $w\,n$ and $x\,o$, will be the Variation of the Velocity $w\,n$ or $\ddot{y}$ in this laſt Moment; and ſo $w\,n - x\,o = \dddot{y}$, the Fluxion of $\ddot{y} = w\,n$, or *third Fluxion* of P M. And after this Manner you may eaſily conceive, how *Fourth*, *Fifth*, &c. *Fluxions are generated.*

905. If the Spaces P p, $p\,q$, $q\,r$, are ſuppoſed equal, and paſſed over in equal Times or Moments, then will the Abſciſs A P flow uniformly, and its momentary Velocities or Fluxions M s, $m\,t$, $n\,v$, *&c.* will be all equal; and conſequently $\dot{x}$ will, in this Caſe, be a conſtant or invariable Quantity; and ſo there can be no $\ddot{x}$, or ſecond Fluxion of A P. But if the Ordinate P M be ſuppoſed to flow with a variable Velocity, then all the Orders of Fluxions will ariſe, as we have ſhewn; and the ſame may be ſhewn of the Arch A M.

906. As to the Rules for operating Second, *&c.* Fluxions, they are the ſame as have been laid down for the *firſt Fluxions* in every Reſpect; as they needs muſt be, ſince nothing more is propoſed in finding any Fluxion, than diſcovering the Ratio of Velocity, with which the Fluent varies its Magnitude, in becoming greater or leſſer: And as this is the ſame Thing in every Kind of Fluent, ſo the Rules for determining it muſt in every Order of Fluxions be the ſame.

907. Hence as the Fluxion of x is $\dot{x}$, ſo that of $\dot{x}$ is $\ddot{x}$; and that of $\ddot{x}$ is $\dddot{x}$; and ſo on.

Note

Note here, that when any of theſe firſt Fluxions are ſuppoſed to be conſtant, they will be expreſſed in *Roman* Characters, as $\dot{\mathrm{x}}, \dot{\mathrm{y}}, \dot{\mathrm{z}}$; but if they are flowing Quantities, they will be repreſented as uſual in *Italics*, as $\dot{x}, \dot{y}, \dot{z}$.

908. Thus the Fluxion of $\dot{\mathrm{x}}\,\dot{y}$ is $\dot{\mathrm{x}}\,\ddot{y}$; and of $\dot{x}\,\dot{y}$, is $\dot{x}\,\ddot{y} + \dot{y}\,\ddot{x}$, and the Fluxion of $\dot{x}\,\dot{y}\,\dot{z}$ is $\dot{x}\,\dot{y}\,\ddot{z} + \dot{x}\,\dot{z}\,\ddot{y} + \dot{y}\,\dot{z}\,\ddot{x}$. Alſo the Fluxion $x\,\dot{y}$ is $x\,\ddot{y} + \dot{y}\,\dot{x}$, and therefore the Fluxion of $x\,\dot{y} + y\,\dot{x}$, is $x\,\ddot{y} + \dot{y}\,\dot{x} + y\,\ddot{x} + \dot{y}\,\dot{x}$, or $x\,\ddot{y} + y\,\ddot{x} + 2\,\dot{x}\,\dot{y}$; which, as it is the firſt Fluxion of $x\,\dot{y} + y\,\dot{x}$, (as that is of xy,) ſo it will be the ſecond Fluxion of xy. And thus the ſecond Fluxion of any other Quantity may be found, as *per* Rule (793).

909. The ſecond Fluxion of any *Power* of any primary Fluxion, is found by the Rule for Powers (794). Thus the Fluxion of $\dot{x}^2$ is $2\,\dot{x}\,\ddot{x}$; and of $\dot{x}^3$, is $3\,\dot{x}^2\,\ddot{x}$. Alſo the Fluxion of $2\,x\,\dot{x}$ (the Fluxion of x^2) is $2\,x\,\ddot{x} + 2\,\dot{x}\,\dot{x}$; which therefore is the ſecond Fluxion of x^2.

910. The Fluxions of *fluxionary Fractions* are alſo found by the proper Rule (801). Thus the Fluxion of $\frac{\dot{x}}{\dot{y}}$ will be found to be $\frac{\ddot{x}\,\dot{y} - \dot{x}\,\ddot{y}}{\dot{y}^2}$; the Fluxion of $\frac{\dot{\mathrm{x}}}{\dot{z}}$, is $\frac{-\,\dot{\mathrm{x}}\,\ddot{z}}{\dot{z}^2}$; of $\frac{1}{\dot{x}}$, is $\frac{-\,\ddot{x}}{\dot{x}^2}$; and of $\frac{y\,\dot{y}}{\dot{x}}$, is $\frac{\dot{x}\,y\,\ddot{y} + x\,\dot{y}^2 - y\,\dot{y}\,\ddot{x}}{\dot{x}^2}$; but if it be $\frac{\mathrm{y}\,\dot{y}}{\dot{x}}$, the Fluxion is $\frac{\mathrm{y}\,\ddot{y} - \mathrm{y}\,\dot{y}\,\ddot{x}}{\dot{x}^2}$; if $\frac{y\,\dot{\mathrm{y}}}{\dot{x}}$, it is $\frac{\dot{y}\,\dot{\mathrm{y}} - y\,\dot{\mathrm{y}}\,\ddot{x}}{\dot{x}^2}$; if $\frac{y\,\dot{y}}{\dot{\mathrm{x}}}$, the Fluxion is $\frac{\dot{\mathrm{x}}\,y\,\ddot{y} + \dot{\mathrm{x}}\,\dot{y}\,\dot{y}}{\dot{\mathrm{x}}^2} = \frac{y\,\ddot{y} + \dot{y}^2}{\dot{\mathrm{x}}}$; and ſo of others.

911. The *ſecond Fluxion* of a *ſurd Quantity* is found in the ſame Manner as the Firſt (798); thus the Fluxion of $\sqrt[2]{\dot{x}} = \dot{x}^{\frac{1}{2}}$, is $\frac{1}{2}\,\dot{x}^{-\frac{1}{2}}\,\ddot{x} = \frac{\ddot{x}}{2\,\dot{x}^{\frac{1}{2}}}$; of $\sqrt[3]{\dot{x}^2} = \dot{x}^{\frac{2}{3}}$, is $\frac{2\,\ddot{x}}{3\,\dot{x}^{\frac{1}{3}}}$; the Fluxion of $\frac{\dot{x}}{2\,x^{\frac{1}{2}}}$ (which is itſelf the Fluxion of $\sqrt[2]{x}$) is $\frac{2\,\ddot{x}\,x^{\frac{1}{2}} - x^{-\frac{1}{2}}\,\dot{x}^2}{4\,x}$, which therefore is the *ſecond Fluxion* of $\sqrt[2]{x}$, and which, if we divide by $4\,x$, will be thus expreſſed, $\frac{1}{2}\,x^{-\frac{1}{2}}\,\ddot{x} - \frac{1}{4}\,x^{-\frac{3}{2}}\,\dot{x}^2$.

912. We have ſhewn that the Fluxion of the Logarithm of

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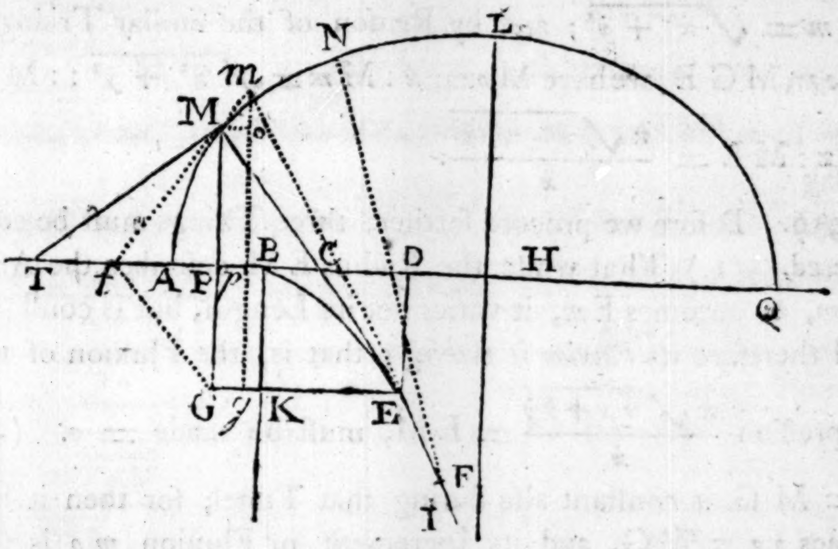
$1 + x$ is $\frac{\dot{x}}{1+x}$, and the Fluxion of this is $\frac{\ddot{x} + \ddot{x}x - \dot{x}\dot{x}}{1+2x+xx}$, which is therefore the ſecond Fluxion of the Logarithm of that Quantity: And thus I preſume the *Second*, *Third*, &c. *Fluxions* of any Quantity propoſed, may be eaſily found by any one who underſtands the Rules for the *firſt Fluxions*, without any further Directions.

CHAP. XVIII.

Of Involute *and* Evolute Curves; *and the Method of inveſtigating the* Radii *of* Evolution, *or* Curvature.

913. BY the Problems of this Chapter, it will appear how great and extenſive the Uſe of the Doctrine of Fluxions is, both of the *Firſt and Second Orders*, ſince their Solutions are hereby rendered facile and perſpicuous, which by the *common Geometry* were not only difficult and tedious, but, to an ordinary Genius, quite inacceſſible, or inſuperable. Theſe relate to the Nature of *Involute and Evolute Curves*, and the Methods of finding the *Radii of Evolution, or Curvature.*

914. If in the Point A, in the Right Line A H, we ſuppoſe one End of a flexible String, or Thread to be faſtened; and Part thereof to lie along, or be coincident with the ſaid Line from A to B, and the other Part to be ſtretched along the Curve B E F, and there fixed at I. And then if the other End A be looſened, and raiſed from its Poſition on the Line and Curve, and moved tight and ſtrait along, while it diſengages itſelf from the Curve B E I, then the Point A will deſcribe a Curve A M N, which is called an *Involute Curve.* And the Curve B E I, is called the *Evolute* of the Curve A M N, or the *Evolute Curve.* And the ſtrait Lines E M, F N, are called the *Radii of Evolution, or Curvature* for the Points M and N.



915. It is evident (since the Length of the Thread A B E I continues the same) that the Part of the Evolute E F is equal to the Difference of the *Radii* F N and E M or that E F = F N — E M. Thus also B E = E M — A B. And when A B = *o*, the Parts of the Curve B E, B F will be equal to their respective *Radii* E M and F N.

916. 'Tis moreover evident, that each Point of the Curve A M N is described by a *different Radius*, which is perpendicular to it in that Point. Thus the Points A, M, N, are described with the different *Radii*, A B, E M, F N, on the respective Centers B, E, F, and consequently the Curvature in those Points will be reciprocally as the Length of those *Radii*.

917. Therefore the *Radii* belonging to each Point of the *Involute* A M N, by their various Intersections form the Curve of the *Evolute* B E F. Thus, suppose the two Points M and *m* contiguous, their *Radii* touch each other in the Point E, and if continued out, would there intersect each other. Therefore,

918. It is a *general Problem, The Nature of the Curve* A M N *being given, to find the Length of the Radius of Evolution* E M, *belonging to a given Point* M.

Let the Ordinate P M, *p m*, be continued out, and from the Point E of the Intersection of the *Radii*, to the Points M and *m*, draw E G perpendicular to the Ordinates, and draw E D parallel to P G; and *m o* to A B. Let A P = *x*, P M = *y*, and M G = *z*, and we have M *o* = $\dot{x}$, *m o* = $\dot{y}$, and

$Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$; and by Reason of the similar Triangles Mom, MGE; we have $Mo = \dot{x} : Mm = \sqrt{\dot{x}^2 + \dot{y}^2} :: MG = z : ME = \frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$.

919. Before we proceed farther, three Things must be considered. (1.) That while the Radius EM describes the Arch Mm, or becomes Em, it varies not its Length, but is constant, and therefore its *Fluxion is nothing*; that is, the Fluxion of the Expression $\frac{z\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}} = EM$, must be made $= o$. (2.) $z = MG$ is constant also during that Time; for then it becomes $og = MG$, and its Increment or Fluxion mo is the Fluxion of PM, that is, $\dot{z} = \dot{y}$. (3.) Since the Fluxion of the Abscifs results from the Motion of the Thread, which we suppose to be equable and uniform, 'tis plain it will be constant, that is $Mo = \dot{x}$, will be the same in all equal Spaces of Time.

920. Therefore making $\dot{x}$ constant, and the Fluxion of $EM = \frac{z\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}$, *viz.* $\dot{z} \times \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} + \frac{z\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} = \frac{\dot{z}\dot{x}\dot{x} + \dot{z}\dot{y}\dot{y} + z\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = o$, which dividing by $\dot{x}$, and putting $\dot{z} = \dot{y}$, (919.) we have $\frac{\dot{y}\dot{x}^2 + \dot{y}^3 + z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}\dot{x}+\dot{y}}} = o$, that is, $\dot{y}\dot{x}^2 + \dot{y}^3 + z\dot{y}\ddot{y} = o$, and so $\dot{x}^2 + \dot{y}\dot{y} = -z\ddot{y}$, and thus we get $z = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}} = MG$. Hence a Perpendicular to the Point G, will intersect the Tangent MC in E, which will give ME, the Radius of Evolution required.

921. *To find the Radius of Evolution for any Point* M, *in the Curve of a Parabola* AMN.

The Equation of the Curve is $px = y^2$, which in Fluxions gives $\dot{y} = \frac{p\dot{x}}{2y} = \frac{p\dot{x}}{2\sqrt{px}}$, and so $\ddot{y} = \frac{-2p\dot{x}\dot{y}}{4yy} = \frac{-2p\dot{x}\dot{y}}{4px} = \frac{-2p\dot{x}}{4px} \times \dot{y} = \frac{-2p\dot{x}}{4px} \times \frac{p\dot{x}}{2\sqrt{px}} = \frac{-p\dot{x}^2}{4x\sqrt{px}}$. Now

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substituting these Values of $\dot{y}$ and $\ddot{y}$, in the general Expression $\frac{\dot{x}\dot{x}+\dot{y}\dot{y}}{-\ddot{y}}=MG$, we shall get an Expression, consisting only of the given Quantities p and x, in the Manner following; $\dot{y}\dot{y}=\frac{p^2\dot{x}^2}{4yy}=\frac{p^2\dot{x}^2}{4px}=\frac{p\dot{x}^2}{4x}$; therefore $\dot{x}\dot{x}+\dot{y}\dot{y}=\dot{x}\dot{x}+\frac{p\dot{x}\dot{x}}{4x}=\frac{4\dot{x}^2x+4xp\dot{x}}{4x}$, this divided by $-\ddot{y}=\frac{p\dot{x}^2}{4x\sqrt{px}}$, will give $\frac{4x\sqrt{px}}{p}+\sqrt{px}=MG=PG+PM$, whence $PG=\frac{4x\sqrt{px}}{p}=DE$.

922. Let tM touch the Parabola in M, then is $tP=2x$, and $PC=\frac{1}{2}p$; for $tP\ (2x):PM\ (y)::PM\ (y):PC=\frac{yy}{2x}$, therefore $2x\times PC=yy=px$, consequently $PC=\frac{1}{2}p$. And as $PM\ (y):PC\ (p\frac{1}{2})::MG\left(\frac{4x\sqrt{px}}{p}+\sqrt{px}\right):GE=PD=\frac{1}{2}p+2x=PC+CD$; but $PC=\frac{1}{2}p$, therefore $CD=2x=Pt$.

923. Now since $PC=\frac{1}{2}p$, is an invariable Quantity, therefore when the Point M coincides with the Vertex A, the Point P will be there too, and consequently PC will then become $AB=\frac{1}{2}p$. *Hence the Vertex* B *of the Evolute* BEF, *touches the Axis of the Parabola, at the Distance of* $AB=\frac{1}{2}p$, *from the Vertex* A.

924. Hence the *Radius of Evolution* in the *Parabola*, when a *Minimum*, is $=AB=\frac{1}{2}p=$ Half the Parameter, consequently the *Curvature of the Parabola* is then and there a *Maximum*, and *equal to the Curvature of a Circle described with the Radius* $AB=\frac{1}{2}p$.

925. To find the Curve (or Nature) of the Evolute BEI.

We have already seen that $BK=PG=\frac{4x\sqrt{px}}{p}$; and $KE=BD=AP+PD-AB=x+\frac{1}{2}p+2x-\frac{1}{2}p=3x$. Let the Abscis $BK=x$, and the Ordinate $KE=y$; then

$x : y :: \frac{4x\sqrt{px}}{p} : 3x$, whence $3xx = \frac{4xy\sqrt{px}}{p}$. But $y = 3x$, and so $x = \frac{y}{3}$, which substituted for x in the Equation above, there comes out $\frac{27}{16}pxx = y^3$, which shews the Evolute B E I is a *second cubical Parabola*, whose Parameter is $\frac{27}{16}$ of the Parameter of the Involute Parabola A M N.

926. Hence it appears, that the *Evolute* is a *Geometrical Curve* when the *Involute* is such; or an Equation may be found to express the Relation of its Abscisses B K, to its Ordinates K E in finite Terms, and free from Fluxions, when the Involute has such an Equation.

927. Having found the *Radius of Evolution* M E, we rectify the Part of the Evolute B E with ease. For it will always be, the Curve B E $=$ M E — A B, *viz. the Curve* B E *will always be equal to the Difference of two given right Lines.*

928. We should next have proceeded to the *Ellipsis* and *Hyperbola*, but that in our Method we must have Recourse to the Radius of Evolution directly M E, which because of the similar Triangles M *o m*, M G E, will be found $= \frac{\overline{\dot{x}^2 + \dot{y}^2} \times \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$; which will prove too troublesome; and therefore, as well as for Ease as Variety, we will investigate another Expression for the Radius M E, in a different Manner than before.

929. In order to this, let M G $= v$, and the Arch A M $= z$; the Rest as before. Then $\dot{x} : \dot{z} :: v : \frac{v\dot{z}}{\dot{x}} =$ M E, which as it is constant, its Fluxion is nothing, therefore the Fluxion of $\frac{v\dot{z}}{\dot{x}}$ is $\frac{\dot{v}\dot{z}\dot{x} + \ddot{z}v\dot{x} - v\dot{z}\ddot{x}}{\dot{x}^2} = 0$. Therefore $\dot{v}\dot{z}\dot{x} = v\dot{z}\ddot{x} - v\dot{x}\ddot{z}$; and so $v = \frac{\dot{v}\dot{z}\dot{x}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = \frac{\dot{z}\dot{x}\dot{y}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}}$, because $\dot{v} = \dot{y}$.

930.

930. Therefore $\dot{x} : \dot{z} :: \frac{\dot{x}\dot{y}\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} : \frac{\dot{y}\dot{z}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = ME$; and $\dot{x} : \dot{y} :: \frac{\dot{x}\dot{y}\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} : \frac{\dot{y}^2\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = GE = PD$, and so $\frac{\dot{z}\dot{y}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} + x = PD + AP = AD$.

Lastly, $\frac{\dot{x}\dot{y}\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} - y = MG - AP = PG = BK$. In all these Expressions, when either $\dot{x}$, $\dot{y}$, or $\dot{z}$, are constant, they may have Unity placed in their Stead, and the Terms when in $\ddot{x}$, $\ddot{z}$, $\ddot{y}$, are found, will vanish, and so the above Expressions will be abridged very considerably. All which will be plain by the following Examples.

931. Let AM be an Arch of the Curve of an *Ellipsis*, or *Hyperbola*, *to find the Radius of Evolution* EM. The Equation for these Curves being $\frac{aa}{bb}yy = ax \mp xx$, (where a and b are the transverse and conjugate Diameters,) we have $y = \frac{b}{a}\sqrt{ax \mp xx}$; and $\dot{y} = \frac{ab \mp 2bx}{2a\sqrt{ax \mp x^2}}$; and $-\ddot{y} = \frac{ba}{4 \times \overline{ax \mp xx}^{\frac{3}{2}}}$. Now by substituting these Values of y, $\dot{y}$, $\ddot{y}$, into the general Equations above, we shall have the Radius of Evolution, and the other Parts expressed by them.

932. Thus $\frac{\dot{y}\dot{z}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = ME$, by making $\dot{x} = 1$, or $\ddot{x} = 0$, will become $\frac{\dot{y}\dot{z}^2}{-\ddot{z}}$; and since $\dot{z} : \dot{y} :: \ddot{y} : \ddot{z}$* $= \frac{\dot{y}\ddot{y}}{\dot{z}}$, therefore $\frac{\dot{y}\dot{z}^2}{-\ddot{z}} = \frac{\dot{z}^3}{-\ddot{y}}$. But $\frac{\dot{z}^3}{-\ddot{y}} = \frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$, in which Equation the Values of $\dot{y}$, and $-\ddot{y}$, above found, being substituted will give the Value of the Radius of Evolution ME.

933. Again, $\frac{\dot{z}\dot{y}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} + x = AD$, becomes $\frac{\dot{z}\dot{y}^2}{-\ddot{z}} + x = \frac{\dot{z}^2\dot{y}}{-\ddot{y}}$

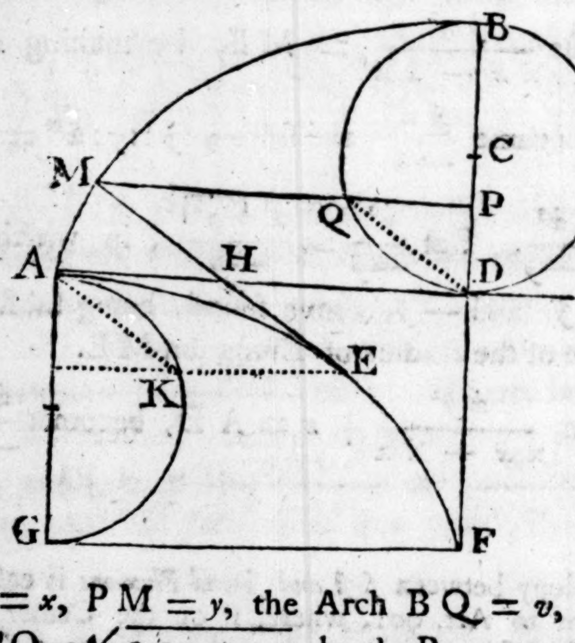
* This Analogy between *first and second Fluxions* is easily deduced from the Figure to Art. 901, where, if on the Center m, a small Arch be described from n to $m\,w$, it will constitute a *second fluxionary Triangle*, similar to the first $m\,t\,w$, which give the Analogy above.

$\frac{\dot{x}^2 \dot{y}}{-\ddot{y}} + x = \mp \frac{a}{2} + \frac{\overline{b^2 \mp a^2 + \overline{a \mp 2x}^3}}{2a^4} =$ AD; this, if we put $x = 0$, will become $\frac{bb}{2a} =$ AB, that is, when the Point M coincides with the Vertex A, the Radius of Evolution then becomes a *Minimum*, and equal to AB = half the Parameter of the Axis AQ.

934. If this Quantity $\frac{bb}{2a}$ be taken from the above Expression for AD, it will leave $\frac{\mp a^2 - b^2}{2} + \frac{\overline{b^2 \mp a^2 \times \overline{a \mp 2x}^3}}{2a^4}$ = KE = BD; if here $b = a$, the Ellipsis will become a Circle, and consequently the Points E and D will ever coincide with the Point B; and therefore the Evolute BE of a Circle will degenerate into a Point, *viz.* the *Center* thereof.

935. If in the Expression gotten for ME, as above directed, we make $x = \frac{1}{2}a =$ AH, that Expression will become = $\frac{aa}{2b}$ = half the Parameter of the conjugate Axis, and the Radius of Evolution, is then a *Maximum*, and the Curvature of the Ellipsis a *Minimum*.

936. *To find the Evolute* AEF, *and Radius of Evolution* ME *of the common Cycloid* AMB.



Let BP = x, PM = y, the Arch BQ = v, and BD = $2a$; then PQ = $\sqrt{2ax - xx}$, by the Property of the Circle; and

and $PM = y = v + \sqrt{2ax - xx}$, by the Property of the Cycloid *. This last Equation in Fluxions will be $\dot{y} = \dot{v} + \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$; but $\dot{v} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$, (875.) therefore $\dot{y} = \frac{2a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}} = \dot{x}\sqrt{\frac{2a - x}{x}}$, and $\ddot{y} = \frac{-a\dot{x}^2}{x\sqrt{2ax - xx}}$, by making $\dot{x}$ invariable. Therefore substituting these Values of $\dot{y}$ and $\ddot{y}$ in the general Expression for $ME = \frac{\overline{\dot{x}^2 + \dot{y}^2} \times \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$ $= \frac{\overline{1 - \dot{y}^2}\sqrt{1 - \dot{y}^2}}{-\ddot{y}}$, we shall get $ME = 2\sqrt{4aa - 2ax}$ $= 2DQ = 2MH$.

937. If we put $x = o$, we have $ME = 2\sqrt{4aa} = 4a$ when the Point M coincides with the Vertical Point B. But if $x = 2a$, then $ME = o$, when the Point M coincides with A. Therefore the Point A is the Beginning of the Evolute, and the Point F the End, and since $FB = 4a$, FD is $= 2a = DB$. Compleat the Rectangle DG; describe the Semicircle AKG; and draw AK parallel to ME or DQ. Then since the Angle $DAK = ADQ$, the Arches $AK = DQ$, and the Chord $AK = DQ$; whence KE is parallel and equal to AH, and therefore equal to the Arch DQ, as is evident from the Generation of the Cycloid; therefore, also EK is equal to the Arch AK, which is the noted Property of the Cycloid. *Therefore the Evolute* AEF *is a Semi-cycloid every alike and equal to the Involute* AMB, *but in an inverted Position.*

Thus I have at Length finished what I have thought necessary to premise, as an Introduction to the mathematical Sciences, the fundamental Principles of Arithmetic, Logarithms, Algebra, Geometry, Conic Sections and Fluxions, in nearly one Thousand Articles. This System of mathematical Elements will, if impartially considered, be found a shorter and more easy Introduction to the Practice of those Arts which depend on them, than any other that has been hitherto published,

* This we shall demonstrate, when we give an Account of the *Genesis* and *Properties* of *mechanical*, or *transcendent Curves*, in our Introduction to *Mechanical Philosophy*.

lished, that I know of. The Transition from one Branch to the other is natural, and shews their mutual Connection and Dependancy, and the Reason of the subsequent Articles evidently appears from the preceding ones: Nor do I know, that I have any where advanced one single Position, that has not been previously demonstrated. Such a Series of preliminary Principles appears to me absolutely necessary, to qualify every Learner for a rational understanding of the most useful Sciencies that are now to follow, and which we can now communicate in a more compendious, and less expensive Form, and in a pleasanter Manner, than we could have done any other Way. If these Institutions are thought to be dry or useless, it must be owing to a Want of Genius, or a natural Incapacity or Disinclination to such Studies. We do not here treat of Trifles, and what we write is intended, in this Part, for such only, who are *born Mathematicians*. They will have a natural Relish for, and take a peculiar Delight in them. It might have been expected, that a *Table* of *Logarithms* should have been added, but we have judged that unnecessary, as they are almost in every one's Hands, and as we could make no Alteration in that Subject, nor afford them cheaper than they may be bought by themselves; it would be scarce fair, to oblige our Readers to buy the same Things over again; especially, as the *artificial Line of Numbers, Sines and Tangents*, supply their Place, in most practical Cases, by Instruments, which will now very soon follow in their proper Order and Places, among the practical mathematical Sciences, which will be the Subjects of the following Volume, where the Reader will find them treated of in a Method new, and with many Alterations and Improvements, both in the Subject Matter, and the Instruments for Practice.

The End of the FIRST VOLUME.

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